Test n° 2 - Answers and Solutions

**Question about course 1.** The subspaces $E_1$ and $E_2$ are supplementary in $V$ if they satisfy the following conditions

i) $V = E_1 + E_2$, that is $\forall v \in V$, $\exists (v_1, v_2) \in E_1 \times E_2$ / $v = v_1 + v_2$

ii) $E_1 \cap E_2 = \{0_V\}$

In this case one can write $V = E_1 \oplus E_2$, that is $\forall v \in V$, $\exists (v_1, v_2) \in E_1 \times E_2$ / $v = v_1 + v_2$

**Problem 2. a)**

\[
P(X) = X^4 - 4X^3 + 9X^2 - 20X + 20
\]

\[
P(2) = 16 - 32 + 36 - 40 + 20 = 0
\]

\[
P'(X) = 4X^3 - 12X^2 + 18X - 20
\]

\[
P'(2) = 32 - 48 + 36 - 20 = 0
\]

\[
P^{(2)}(X) = 12X^2 - 24X + 18
\]

\[
P^{(2)}(2) = 48 - 48 + 18 = 18 \neq 0
\]

Then 2 is a root of $P$ of multiplicity 2.

b) (1) For every $t \in \mathbb{R}$, we have

\[
P(it) = i^4 t^4 - 4i^3 t^3 + 9i^2 t^2 - 20it + 20
\]

\[
= t^4 - 4t^3 + 9t^2 - 20t + 20
\]

\[
= (t^4 - 9t^2 + 20) + (4it^3 - 20it)
\]

\[
= (t^4 - 9t^2 + 20) + 4it(t^2 - 5)
\]

(2) By factorizing real and imaginary parts of $P(it)$, we get

\[
\Re(P(it)) = t^4 - 9t^2 + 20 = (t^2 - 4)(t^2 - 5) = (t - \sqrt{4})(t + \sqrt{4})(t - \sqrt{5})(t + \sqrt{5})
\]

\[
\Im(P(it)) = 4t(t^2 - 5) = 4t(t - \sqrt{5})(t + \sqrt{5})
\]

Consequently

\[
P(it) = 0 \iff \begin{cases} 
\Re(P(it)) = 0 \\
\Im(P(it)) = 0
\end{cases}
\]

\[
\iff \begin{cases} 
t = \sqrt{4} \quad \text{or} \quad t = -\sqrt{4} \\
t = \sqrt{5} \quad \text{or} \quad t = -\sqrt{5}
\end{cases}
\]

\[
\iff t = \sqrt{5} \quad \text{or} \quad t = -\sqrt{5}
\]

Equivalently, $it_0$ and $-it_0$ with $t_0 = \sqrt{5} > 0$ are roots of $P$. 
c) We got three roots of \( P \), one of which is of multiplicity 2. Consequently \( P \) may be written as
\[
P(X) = (X - 2)^2(X - it_0)(X + it_0)Q(X)
\]
Moreover \( P \) and \((X - 2)^2(X - it_0)(X + it_0)\) are of degree 4, then \( Q(X) \) is of degree 1, that is a constant polynomial equal to the leading coefficient of \( P \) which is 1. Finally we have
\[
P(X) = (X - 2)^2(X - it_0)(X + it_0) = (X - 2)^2(X^2 + t_0^2)
\]

Problem 3. a) (1) We have
\[
\begin{align*}
x > 0 \quad & \Rightarrow \quad \ln(x) \in \mathbb{R} \\
y > 0 \quad & \Rightarrow \quad \ln(y) \in \mathbb{R} \\
\Rightarrow \quad & \ln(x) \ln(y) \in \mathbb{R} \\
\Rightarrow \quad & x * y = e^{\ln(x)\ln(y)} > 0
\end{align*}
\]
That is * is a binary operation on \([0, +\infty[\).

(2) Since the multiplication of real numbers is commutative, we get for every positive real numbers \( x \) and \( y \):
\[
x * y = e^{\ln(x)\ln(y)} = e^{\ln(y)\ln(x)} = y * x
\]
Then * is commutative.

(3) For every positive real numbers \( x, y \) and \( z \), we have
\[
\begin{align*}
(x * y) * z &= e^{\ln(x)\ln(y)} * z = e^{\ln(e^{\ln(x)\ln(y)})\ln(z)} = e^{\ln(x)\ln(y)\ln(z)} \\
x * (y * z) &= x * e^{\ln(y)\ln(z)} = e^{\ln(x)\ln(e^{\ln(y)\ln(z)})} = e^{\ln(x)\ln(y)\ln(z)}
\end{align*}
\]
Consequently \((x * y) * z = x * (y * z)\) that is * is associative.

b) For every \( x > 0 \), we have
\[
\begin{align*}
e * x &= e^{\ln(e)\ln(x)} = e^{\ln(x)} = x \quad \text{since } \ln(e) = 1 \\
x * e &= e * x = x \quad \text{since * is commutative}
\end{align*}
\]
Then \( e \) is the identity element for *.

c) (1) For every \( x \in E = ]0, +\infty[ - \{1\} \), we have
\[
x * y = y * x = e \Rightarrow e^{\ln(x)\ln(y)} = e^1 \Rightarrow \ln(x) \ln(y) = 1 \Rightarrow \ln(y) = \frac{1}{\ln(x)} \Rightarrow y = e^{\frac{1}{\ln(x)}}
\]
and \( y \) is well defined because \( x \neq 1 \Rightarrow \ln(x) \neq 0 \). Moreover \( y = e^{\frac{1}{\ln(x)}} > 0 \) and \( y \neq 1 \) because \( \frac{1}{\ln(x)} \neq 0 \).
Consequently \( x \in E \) has an inverse element \( x^{-1} \in E \) which is \( x^{-1} = e^{\frac{1}{\ln(x)}} \).

(2) Finally \((E, *)\) is an abelian group.