Test n° 1  
(1 hour)

**Question about course 1.** Recall the axioms that a ring $(R, +, \times)$ must satisfy.

**Exercise 2.** Prove by induction the following claim for every integer $n \in \mathbb{N}$

$$6^{2n+1} + 2^{3n+1} \equiv 1 \ [7]$$

**Problem 3.** Consider the following set

$$A = \left\{ a + b\sqrt{2} / (a,b) \in \mathbb{Z}^2 \right\}$$

1. Assuming $\sqrt{2} \notin \mathbb{Q}$ (that is there exist no integers $p$ and $q$ such that $\sqrt{2} = \frac{p}{q}$), show that

$$\forall a + b\sqrt{2} \in A, \ a + b\sqrt{2} = 0 \iff a = b = 0$$

2. Prove that $(A, +, \times)$ is a commutative unital ring but not a field.

3. Consider the following maps

$$\varphi : A \rightarrow A \quad \text{and} \quad N : A \rightarrow A \quad \text{where} \quad x \mapsto x \times \varphi(x)$$

(a) Prove that

$$\left\{ \begin{array}{ll} \forall x \in A, & N(x) \in \mathbb{Z} \\ \forall (x,y) \in A^2, & N(xy) = N(x)N(y) \end{array} \right.$$  

(b) Deduce that

$$\forall x \in A, \ x \text{ has a multiplicative inverse in } A \iff N(x) = 1 \text{ or } -1$$

**Problem 4.** Consider the following map for some parameter $\lambda \in \mathbb{R}^*$

$$f_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{where} \quad (x,y) \mapsto \left(\lambda x, \frac{y}{\lambda}\right)$$

1. Prove that $f_\lambda$ is a group endomorphism from $(\mathbb{R}^2, +)$ to itself.

2. Consider the following set

$$F = \{ f_\lambda / \lambda \in \mathbb{R}^* \}$$

(a) Determine $f_\lambda \circ f_\mu$ for any $\lambda$ and $\mu$ in $\mathbb{R}^*$.

(b) Deduce that $\circ$ is a binary operation on $F$.

(c) Prove that $(F, \circ)$ is a group.

3. Consider the following map

$$h : \mathbb{R}^* \rightarrow F \quad \text{where} \quad \lambda \mapsto f_\lambda$$

(a) Prove that $h$ is a group isomorphism from $(\mathbb{R}^*, \times)$ to $(F, \circ)$.

(b) Deduce that $(F, \circ)$ is an abelian group.