Construction de fractions rationnelles à dynamique prescrite

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Construction of rational maps with prescribed dynamics

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Field: Study of holomorphic dynamical systems

Motivation: Find some examples of rational maps with particular complicated dynamics

Questions: 1- How to construct rational maps from dynamical informations ?
2- Which kind of rational maps is it possible to construct ?

Main tools: Quasiconformal surgery and Thurston theory
Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$.

For every $z_0 \in \hat{\mathbb{C}}$, consider its forward orbit $\{ z_n = f^n(z_0) \mid n \geq 1 \}$.

\[
\begin{array}{ccccccc}
Z_0 & \mapsto & Z_1 & \mapsto & Z_2 & \mapsto & Z_3 & \mapsto & \ldots
\end{array}
\]
Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$.

For every $z_0 \in \hat{\mathbb{C}}$, consider its forward orbit $\{z_n = f^{\circ n}(z_0) / n \geq 1\}$.

\[
\begin{align*}
  z_0 & \mapsto z_1 & f \\
  z_1 & \mapsto z_2 & f \\
  z_2 & \mapsto z_3 & f \\
  & \cdots & & \ldots
\end{align*}
\]

**Definition (Fatou and Julia sets)**

- the Fatou set is

  \[ \mathcal{F}(f) = \{ z_0 \in \hat{\mathbb{C}} / (f^{\circ n})_{n \geq 1} \text{ is a normal family at } z_0 \} \]

- the Julia set is

  \[ \mathcal{J}(f) = \hat{\mathbb{C}} - \mathcal{F}(f) \]
Theorem

Let $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. $\mathcal{J}(f)$ is a nonempty fully invariant closed and perfect set. Furthermore either

- $\mathcal{J}(f)$ is connected,
- or else $\mathcal{J}(f)$ has uncountably many connected components.
Theorem

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. $\mathcal{J}(f)$ is a nonempty fully invariant closed and perfect set. Furthermore either

- $\mathcal{J}(f)$ is connected,
- or else $\mathcal{J}(f)$ has uncountably many connected components.

Example

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Theorem

Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map of degree \( d \geq 2 \).
If there exists an attracting fixed point \( z_\infty \) of \( f \) such that every critical point of \( f \) lies in the immediate attracting basin of \( z_\infty \) then

\[
\exists \text{ a homeomorphism } \phi /
\]

\[
\begin{align*}
\mathcal{J}(f) & \xrightarrow{f} \mathcal{J}(f) \\
\downarrow \phi & \quad \downarrow \phi \\
\Sigma_d & \xrightarrow{\sigma} \Sigma_d
\end{align*}
\]

where

- \( \Sigma_d = \{1, 2, \ldots, d\}^\mathbb{N} \) is a Cantor set
- \( \varepsilon = (\varepsilon_0 \varepsilon_1 \varepsilon_2 \ldots) \mapsto \sigma(\varepsilon) = (\varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots) \) is the shift map
Example (McMullen)
Example (McMullen)

**Theorem**

\[ f_{CoC} \text{ acts on } \mathcal{J}_{CoC} = \{J \text{ Julia component of } \mathcal{J}(f_{CoC})\} \approx \bigcup_{\alpha \in \Sigma_2} C_\alpha. \]

\[ \exists \text{ a homeomorphism } \phi/ \]

\[ \mathcal{J}_{CoC} \xrightarrow{f_{CoC}} \mathcal{J}_{CoC} \]

\[ \Sigma_2 \xrightarrow{\sigma} \Sigma_2 \]

\[ \mathcal{J}_{CoC} \xrightarrow{f_{CoC}} \mathcal{J}_{CoC} \]

\[ \phi \]

\[ \Sigma_2 \xrightarrow{\sigma} \Sigma_2 \]

\[ \phi \]

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Example (McMullen)
Example (McMullen)
Example (McMullen)
Theorem

\[ \exists \textit{ a homeomorphism } \varphi / \]

\[ \mathcal{J}_{\text{CoC}} \xrightarrow{f_{\text{CoC}}} \mathcal{J}_{\text{CoC}} \]

\[ \varphi \downarrow \downarrow \]

\[ \mathcal{J}_{\text{C}} \xrightarrow{\tau_{\text{C}}} \mathcal{J}_{\text{C}} \]

where

- \( \tau_{\text{C}} : [0, 1] \rightarrow [0, 1], x \mapsto \begin{cases} 
3x & \text{if } x \in [0, \frac{1}{2}] \\
3(1 - x) & \text{if } x \in [\frac{1}{2}, 1]
\end{cases} \)

- and \( \mathcal{J}_{\text{C}} = \{ x \in [0, 1] / \forall n \geq 0, \tau_{\text{C}}^n(x) \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \} \)
Consider $P_c : z \mapsto z^2 + c$ where $c \approx -0.157\ldots + 1.032\ldots i$

Let $\mathcal{J}_H$ be the intersection between $\mathcal{J}(P_c)$ and the Hubbard tree $\mathcal{H}$
Theorem (Persian carpet)

There exists a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

\[
\exists \text{ a homeomorphism } \varphi / \quad \mathcal{J}_H(f) \xrightarrow{f} \mathcal{J}_H(f)
\]

where $\mathcal{J}_H(f)$ is a subset of Julia components of $f$.

Moreover,

- there exists only one fixed Julia component $J_\alpha$
- $\forall J \in \mathcal{J}_H(f) - \bigcup_{n \geq 0} (f^n)^{-1}(J_\alpha)$, $J$ is a Jordan curve
Figure: A Persian carpet
Consider the following abstract Hubbard tree $\mathcal{H} = (T, \tau)$. 
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We equip the Hubbard tree $H$ with a weight function $w$. 

![Hubbard tree diagram]
We equip the Hubbard tree $\mathcal{H}$ with a weight function $w$.

Fact (the weighted Hubbard tree $(\mathcal{H}, w)$ is unobstructed)

$$
\begin{align*}
\tau(e_{\alpha,c_0}) &= e_{\alpha,c_1} \\
\tau(e_{\alpha,c_1}) &= e_{\alpha,c_2} \\
\tau(e_{\alpha,c_2}) &= e_{\alpha,c_0} \cup e_{c_0,c_3} \\
\tau(e_{c_0,c_3}) &= e_{\alpha,c_1} \cup e_{\alpha,c_0}
\end{align*}
$$

gives

$$
M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
1 & 1 & 0 & 0
\end{pmatrix}
$$

with

$$
\lambda(\mathcal{H}) := \lambda(M) \approx 0.918 < 1
$$
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Question: How to construct a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ “encoded” by the unobstructed weighted Hubbard tree $(\mathcal{H}, w)$?
Question: How to construct a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ “encoded” by the unobstructed weighted Hubbard tree $(\mathcal{H}, w)$ ?

Answer: By quasiconformal surgery!
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\[
\begin{align*}
&c_1 \xrightarrow{2} c_2 \xrightarrow{2} c_3 \\
&1 \xleftarrow{1}
\end{align*}
\]
\[ c_1 = 1 \xrightarrow{2} c_2 = \infty \xrightarrow{2} c_3 = 0 \]

\[ \hat{f} = (z \mapsto z^2) \circ \left( z \mapsto \frac{1}{1-z} \right) = \left( z \mapsto \frac{1}{(1-z)^2} \right) \]
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\[ c_1 = 1 \xrightarrow{2} c_2 = \infty \xrightarrow{2} c_3 = 0 \]

\[ \hat{f} = (z \mapsto z^2) \circ \left( z \mapsto \frac{1}{1-z} \right) = \left( z \mapsto \frac{1}{(1-z)^2} \right) \]
Step 1 - Cutting off

Lemma (equipotentials layout)

Given any positive constant $C > 0$, there exist five equipotentials $\beta_0, \beta_1, \beta_2, \gamma_-3$ and $\gamma_+3$ such that

(i) $\beta_0 \subset B(0)$, $\beta_1 \subset B(1)$ and $\beta_2 \subset B(2)$

(ii) $\gamma_-3, \gamma_+3 \subset B(0)$ and $|\phi_0(\beta_0)| > |\phi_0(\gamma_-3)| > |\phi_0(\gamma_+3)|$

(iii) the following inequalities hold

\[
\begin{aligned}
\mod(\alpha_1, \beta_1) &< \mod(\alpha_0, \beta_0) \\
\frac{1}{2} \mod(\alpha_2, \beta_2) &< \mod(\alpha_1, \beta_1) \\
\frac{1}{2} \mod(\alpha_0, \beta_0) + \frac{1}{2} \mod(\gamma_-3, \gamma_+3) &< \mod(\alpha_2, \beta_2) \\
\mod(\alpha_0, \beta_0) + \mod(\alpha_1, \beta_1) + C &< \mod(\gamma_-3, \gamma_+3)
\end{aligned}
\]

(1)

\[
\frac{1}{2} \mod(\alpha_0, \gamma_+3) < \mod(\alpha_2, \beta_2)
\]

(2)
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Sketch of proof for equipotentials layout Lemma.

Compare

\[
\begin{align*}
\text{mod}(\alpha_1, \beta_1) &< \text{mod}(\alpha_0, \beta_0) \\
\frac{1}{2} \text{mod}(\alpha_2, \beta_2) &< \text{mod}(\alpha_1, \beta_1) \\
\frac{1}{2} \text{mod}(\alpha_0, \beta_0) + \frac{1}{2} \text{mod}(\gamma_{-3}, \gamma_{+3}) &< \text{mod}(\alpha_2, \beta_2) \\
\text{mod}(\alpha_0, \beta_0) + \text{mod}(\alpha_1, \beta_1) + C &< \text{mod}(\gamma_{-3}, \gamma_{+3})
\end{align*}
\]

(1)

with

\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
1 & 1 & 0 & 0
\end{pmatrix}
\]

Furthermore \(\lambda(M) < 1\) implies \(\exists x \in \mathbb{R}^4 \mid x > 0\) and \(Mx < x\)
Step 2 - The branching piece

Define

\[ F_{|\hat{\mathbb{C}} - D(0,\beta_0',1)} = \hat{f}_{|\hat{\mathbb{C}} - D(0,\beta_0',1)} \]
Step 3 - Preimage of the branching piece

Lemma (inverse Grötzsch’s inequality - Cui Guizhen and Tan Lei)

\[ \exists C > 0 \land \forall \beta_0, \beta_1, \text{mod}(\beta_1, \beta_0) < \text{mod}(\alpha_0, \beta_0) + \text{mod}(\alpha_1, \beta_1) + C \]

\[ (1) \Rightarrow \text{mod}(\alpha_0, \beta_0) + \text{mod}(\alpha_1, \beta_1) + C < \text{mod}(\gamma_3, \gamma_3) \]
Step 3 - Preimage of the branching piece

Lemma (inverse Grötzsch’s inequality - Cui Guizhen and Tan Lei)

\[ \exists C > 0 / \forall \beta_0, \beta_1, \text{mod}(\beta_1, \beta_0) < \text{mod}(\alpha_0, \beta_0) + \text{mod}(\alpha_1, \beta_1) + C \]

\[ \exists \beta'_{-3,1}, \beta'_{+3,0} \subset A(\gamma_{-3}, \gamma_{+3}) / \text{mod}(\beta'_{-3,1}, \beta'_{+3,0}) = \text{mod}(\beta_1, \beta_0) \]
Define $F$ on $A(\beta'_1, \beta'_0)$ to be a biholomorphic map such that

- $F$ maps $A(\beta'_{-3}, \beta'_{+3})$ onto $A(\beta_1, \beta_0)$
- $F$ extends diffeomorphically to $\overline{A(\beta'_1, \beta'_0)}$ mapping $\beta'_{-3}$ onto $\beta_1$ and $\beta'_{+3}$ onto $\beta_0$
Step 4 - Folding

Define

\[ F|_{A(\beta_0, \gamma_{-3})} = \psi \circ (z \mapsto z + \frac{1}{z}) \circ \varphi \]
Extends quasiregularly $F$ on $\overline{A(\beta'_{0,1}, \beta_0)} \cup \overline{A(\gamma_{-3}, \beta'_{-3,1})}$
Step 5 - End with an end
Let $\delta'_{+3,-3} \subset A(\beta_{+3,0}, \gamma_{+3})$ be a smooth curve.
Define $F$ on $D(\delta'_{+3,-3})$ to be a biholomorphic map such that
- $F$ maps $D(\delta'_{+3,-3})$ onto $D(0, \gamma_{-3})$
- $F$ extends diffeomorphically to $D(\delta'_{+3,-3})$
  mapping $\delta'_{+3,-3}$ onto $\gamma_{-3}$
Define $F$ on $D(0, \gamma_{+3})$ to be any biholomorphic map such that

- $F$ maps $D(0, \gamma_{+3})$ onto $D(p, \zeta) \subset A(\beta_0, \gamma_{-3})$ with $F(0) = p$
- $F$ extends diffeomorphically to $D(0, \gamma_{+3})$
- mapping $\gamma_{+3}$ onto $\zeta$

Extends quasiregularly $F$ on $P(\beta'_{+3,0}, \delta'_{+3,-3}, \gamma_{+3})$
Final Step

- $F$ is holomorphic on an open set $H \subset \hat{\mathbb{C}}$
  
  $$H = \left( \hat{\mathbb{C}} - D(0, \beta'_{0,1}) \right) \bigcup A(\beta'_{-3,1}, \beta'_{+3,0}) \bigcup A(\beta_0, \gamma_{-3})$$
  
  - Step 2
  - Step 3
  - Step 4
  
  $$\bigcup D(\delta'_{+3,-3}) \cup D(0, \gamma_{+3})$$
  
  - Step 5

- $F$ extends quasiregularly to the complement $Q = \hat{\mathbb{C}} - H$
  
  $$Q = A(\beta'_{0,1}, \beta_0) \cup A(\gamma_{-3}, \beta'_{-3,1}) \bigcup P(\beta'_{+3,0}, \delta'_{+3,-3}, \gamma_{+3})$$
  
  - Step 4
  - Step 5

- $\exists$ an open set $A \subset H$ such that $F(A) \subset A$ and $F^\circ 2(Q) \subset A$
  
  $$A = A(\beta_0, \gamma_{-3}) \cup D(1, \beta_1) \cup D(\infty, \beta_2) \cup D(0, \gamma_{+3})$$
Quasiconformal surgery principle: We may apply Morrey-Ahlfors-Bers theorem to get

\[ \exists \text{ a quasiconformal map } \phi \text{ with } F\text{-invariant dilatation} \]

Therefore \( f = \phi \circ F \circ \phi^{-1} \) is a rational map.
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This ramification portrait is realized by \( \hat{f} = \left( z \mapsto \frac{1}{(1 - z)^2} \right) \)
Question: Which kind of ramification portraits is realized by post-critically finite rational maps?
**Question:** Which kind of ramification portraits is realized by post-critically finite rational maps?

**Answer:** The Thurston’s topological characterization!
Theorem (Thurton’s topological characterization)

Let \( f : \mathbb{S}^2 \to \mathbb{S}^2 \) be a ramified covering with \(|P_f| < \infty\).

Then there exists a rational map \( \hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that

\[
\exists \varphi_0, \varphi_1 \text{ homeomorphisms} / \begin{cases} 
(i) & \mathbb{S}^2 \overset{\varphi_1}{\longrightarrow} \hat{\mathbb{C}} \\
(ii) & \mathbb{S}^2 \overset{\varphi_0}{\longrightarrow} \hat{\mathbb{C}} \\
(iii) & \varphi_0(P_f) = \varphi_1(P_f) = \hat{P_f} \\
\end{cases}
\]

if and only if \( f \) has no Thurston obstruction.
Topological part

Definition \((N\text{-cyclic ramification portrait of polynomial type})\)

A ramification portrait \(\mathcal{R} = (\Omega, P, \sigma, \nu)\) is \(N\text{-cyclic ramification portrait of polynomial type}\) if

- \(\mathcal{R}\) is branch compatible: \(\forall y \in P, \sum_{\sigma(x) = y} \nu(x) \leq \deg(\mathcal{R})\)
- \(\exists \infty \in \Omega \cup P / \sigma(\infty) = \infty\) and \(\nu(\infty) = \deg(\mathcal{R})\)
- \(\forall \omega \in \Omega - \{\infty\}, \omega\) is \(\sigma\)-periodic
- \(P - \{\infty\}\) is the union of exactly \(N\) disjoint periodic cycles
Topological part

Definition \((N\)-cyclic ramification portrait of polynomial type)\)

A ramification portrait \(\mathcal{R} = (\Omega, P, \sigma, \nu)\) is **\(N\)-cyclic ramification portrait of polynomial type** if

- \(\mathcal{R}\) is branch compatible: \(\forall y \in P, \sum_{\sigma(x) = y} \nu(x) \leq \deg(\mathcal{R})\)
- \(\exists \infty \in \Omega \cup P / \sigma(\infty) = \infty\) and \(\nu(\infty) = \deg(\mathcal{R})\)
- \(\forall \omega \in \Omega - \{\infty\}, \omega\) is \(\sigma\)-periodic
- \(P - \{\infty\}\) is the union of exactly \(N\) disjoint periodic cycles

Theorem (topological realization)

*Every \(N\)-cyclic ramification portrait of polynomial type is realized by a ramified covering \(f : S^2 \rightarrow S^2\).*
Sketch of proof for topological realization.

$S_1$

$f$

$S_2$
Analytical part

Theorem (polynomial criterion)

If a topological polynomial $f$ has a Thurston obstruction then

(i) $f$ has a Levy cycle $\Gamma$ contained in the Thurston obstruction

(ii) there exist some post-critical points of $f$ whose iterations do not accumulate a critical point
Analytical part

Theorem (polynomial criterion)

If a topological polynomial $f$ has a Thurston obstruction then

(i) $f$ has a Levy cycle $\Gamma$ contained in the Thurston obstruction

(ii) there exist some post-critical points of $f$ whose iterations do not accumulate a critical point

Corollary (Levy’s criterion)

Let $f : S^2 \to S^2$ be a topological polynomial with $|P_f| < \infty$

If every critical point falls into a periodic cycle containing a critical point then $f$ has no Thurston obstruction.
Analytical part

Theorem (polynomial criterion)

If a topological polynomial $f$ has a Thurston obstruction then

(i) $f$ has a Levy cycle $\Gamma$ contained in the Thurston obstruction

(ii) there exist some post-critical points of $f$ whose iterations do not accumulate a critical point

Corollary (Levy’s criterion)

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a topological polynomial with $\left| P_f \right| < \infty$
If every critical point falls into a periodic cycle containing a critical point then $f$ has no Thurston obstruction.

Corollary (analytical realization)

Every $N$-cyclic ramification portrait of polynomial type is realized by a polynomial $\hat{f} : \mathbb{S}^2 \to \mathbb{S}^2$. 
Let $\hat{R}$ be a $N$-cyclic ramification portrait of polynomial type.

\[
\begin{align*}
\hat{\nu}(c_1^1) & \quad \hat{\nu}(c_1^2) & \quad \hat{\nu}(c_1^{n_1-1}) \\
\hat{\nu}(c_2^1) & \quad \hat{\nu}(c_2^2) & \quad \hat{\nu}(c_2^{n_2-1}) \\
\hat{\nu}(c_3^1) & \quad \hat{\nu}(c_3^2) & \quad \hat{\nu}(c_3^{n_3-1}) \\
\vdots & \quad \vdots & \quad \vdots \\
\hat{\nu}(c_N^1) & \quad \hat{\nu}(c_N^2) & \quad \hat{\nu}(c_N^{n_N-1}) \\
\hat{\nu}(c_\infty) & \quad \deg(\hat{R})
\end{align*}
\]
Let $\hat{R}$ be a $N$-cyclic ramification portrait of polynomial type.

\[
\begin{align*}
&c_1^1 \quad \hat{\nu}(c_1^1) \quad c_2^1 \quad \hat{\nu}(c_2^1) \quad \cdots \quad \hat{\nu}(c_{n_1-1}^1) \quad c_{n_1}^1 \\
&\quad \quad \quad \quad \hat{\nu}(c_{n_1}^1) \\
&c_1^2 \quad \hat{\nu}(c_1^2) \quad c_2^2 \quad \hat{\nu}(c_2^2) \quad \cdots \quad \hat{\nu}(c_{n_2-1}^2) \quad c_{n_2}^2 \\
&\quad \quad \quad \quad \hat{\nu}(c_{n_2}^2) \\
&\vdots & & \vdots \\
&c_1^N \quad \hat{\nu}(c_1^N) \quad c_2^N \quad \hat{\nu}(c_2^N) \quad \cdots \quad \hat{\nu}(c_{n_N-1}^N) \quad c_{n_N}^N \\
&\quad \quad \quad \quad \hat{\nu}(c_{n_N}^N) \\
&c_1^{N+1} \quad \hat{\nu}(c_1^{N+1}) \quad c_2^{N+1} \quad \hat{\nu}(c_2^{N+1}) \quad \cdots \quad \hat{\nu}(c_{n_{N+1}-1}^{N+1}) \quad c_{n_{N+1}}^{N+1} \\
&\quad \quad \quad \quad \hat{\nu}(c_{n_{N+1}}^{N+1})
\end{align*}
\]
Let $\mathcal{R}$ be the following ramification portrait.

\[\begin{array}{c}
\nu(p_1) \rightarrow \hat{\nu}(c_1^1) \rightarrow \ldots \rightarrow \hat{\nu}(c_{n_1}^1) \\
p_1 \quad \vdots \quad p_1' \quad \vdots \quad \vdots \quad \vdots
\end{array}\]

\[\begin{array}{c}
\nu(c_{n_1}^1) \\
\vdots \\
\nu(c_{n_m}^m)
\end{array}\]

\[\begin{array}{c}
p_m \nu(p_m) = \hat{\nu}(c_m^m) \rightarrow \hat{\nu}(c_{n_m}^m) \\
p_m \quad \vdots \quad p_m' \quad \vdots \quad \vdots \quad \vdots
\end{array}\]

\[\begin{array}{c}
\nu(c_{n_m}^m) \\
\vdots \\
\nu(c_{n_{m+1}}^{m+1})
\end{array}\]

\[\begin{array}{c}
c_1^{m+1} \rightarrow \ldots \rightarrow \hat{\nu}(c_{n_{m+1}}^{m+1}) \\
c_1^{m+1} \quad \vdots \quad c_1^{m+1} \quad \vdots \quad \vdots \quad \vdots
\end{array}\]

\[\begin{array}{c}
\hat{\nu}(c_{n_{m+1}}^{m+1}) \\
\vdots \\
\hat{\nu}(c_{n_{N+1}}^{N+1})
\end{array}\]

\[\begin{array}{c}
c_1^{N+1} \rightarrow \ldots \rightarrow \hat{\nu}(c_{n_{N+1}}^{N+1}) \\
c_1^{N+1} \quad \vdots \quad c_1^{N+1} \quad \vdots \quad \vdots \quad \vdots
\end{array}\]

\[\begin{array}{c}
\hat{\nu}(c_{n_{N+1}}^{N+1}) \\
\vdots \\
\hat{\nu}(c_{n_{N+1}}^{N+1})
\end{array}\]
Definition (admissible weighted Hubbard tree)

Such a ramification portrait $\mathcal{R}$ may be deduced from a weighted Hubbard tree $(\mathcal{H}, w)$ such that

- **tree shape condition:**
  $\mathcal{H}$ is a starlike tree around an unique branched point $\alpha$, every $p_i$ is the endpoint of two exactly two edges and every $c_k^i$ is an end

- **realization condition:**
  the associated sub-ramification portrait $\hat{\mathcal{R}}$ is a $N$-cyclic ramification portrait of polynomial type

- **Thurston condition:**
  $(\mathcal{H}, w)$ is unobstructed
Theorem (realization of admissible weighted Hubbard tree)

For every admissible weighted Hubbard tree \((\mathcal{H}, w)\)
there exists a rational map \(f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) such that

(i) \(f\) realizes the associated ramification portrait \(\mathcal{R}\)

(ii) the Julia set \(\mathcal{J}(f)\) is disconnected
Sketch of the proof.

**First idea:** Folding

![Diagram showing a mathematical proof concept with symbols and notation describing the folding process.](image)
Sketch of the proof.

First idea: Folding

Second idea: Final Step
Use a result of Cui Guizhen and Tan Lei generalizing the Thurston’s theorem for some non-post-critically finite maps.
Figure: Different motifs of Persian carpets
Figure: Different motifs of Persian carpets
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Figure: Different motifs of Persian carpets
• Enlarge the **tree shape condition** and the **realization condition**.
• Encode the exchanging dynamics of Julia components.
• Extends continuously the encoding map $\pi : \mathcal{J}_\mathcal{H} \rightarrow \mathcal{H}$ to $\hat{\mathbb{C}}$. 
- Enlarge the tree shape condition and the realization condition.
- Encode the exchanging dynamics of Julia components.
- Extends continuously the encoding map $\pi : \mathcal{J}_\mathcal{H} \to \mathcal{H}$ to $\hat{\mathbb{C}}$.

And more generally,
- What about the unicity?
- What about the converse problem?
Merci de votre attention !