Wandering under Bishop’s trees

Sébastien Godillon
Plan

Bishop’s construction 1: Motivation and main results

Bishop’s construction 2: Sketch of the proof by quasiconformal foldings

Existence and non-existence of wandering domains for entire functions

Examples of wandering domains in Eremenko-Lyubich’s class
Wandering under Bishop’s trees

Bishop’s construction 1: Motivation and main results

Sébastien Godillon
Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function with
- exactly two critical values, say $-1$ and $+1$
- no finite asymptotic values

**Question**: What does $f$ “look like”??
$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.
\[ T = f^{-1}([-1, +1]) \] is an infinite bipartite tree.

\[ \cosh: \mathbb{H}_r \to \mathbb{C}\setminus[-1, +1] \] is a universal cover.
$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.

$\forall \Omega$ c.c. of $\mathbb{C} \setminus T$, $\tau|_{\Omega} = (\cosh^{-1} \circ f|_{\Omega}) : \Omega \to \mathbb{H}_r$ is conformal.
Conversely: How to construct $f$ from $(T, \tau)$?
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More precisely, given
- an infinite bipartite tree $T \subset \mathbb{C}$ with “smooth” enough geometry
- a map $\tau$ such that $\tau|_{\Omega} : \Omega \to \mathbb{H}_r$ is conformal, $\forall \Omega$ c.c. of $\mathbb{C} \setminus T$

does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f = \cosh \circ \tau$?
Conversely: How to construct $f$ from $(T, \tau)$?

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does there exist an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f = \cosh \circ \tau$?

Main problem: $\cosh \circ \tau$ is not continuous across $T$ in general.
Solution: Modify \((T, \tau)\) in a small neighborhood \(T(r_0)\) of \(T\).
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More precisely, replace \((T, \tau)\) by \((T', \eta)\) such that

- \(T \subset T' \subset T(r_0)\)
- \(\eta = \tau\) off \(T(r_0)\)
- \(\eta|_{\Omega'} : \Omega' \to \mathbb{H}_r\) is \(K\)-quasiconformal, \(\forall \Omega'\) c.c. of \(\mathbb{C} \setminus T'\)
- \(\cosh \circ \eta\) continuously extends across \(T'\)
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Then apply Morrey-Ahlfors-Bers measurable Riemann mapping theorem:

\[ \exists\ \text{an entire function } f \text{ and a quasiconformal map } \phi \text{ such that } f \circ \phi = \cosh \circ \tau \text{ off } T(r_0) \]
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The neighborhood of $T$

$\forall r > 0$, define an open neighborhood of $T$ as follows

$$T(r) = \bigcup_{e \text{ edge of } T} \left\{ z \in \mathbb{C} / \text{dist}(z, e) < r \text{ diam}(e) \right\}$$
Lemma 0

If $T$ has bounded geometry, namely $\exists M > 0$ such that

1. edges of $T$ are $C^2$ with uniform bounds
2. angles between adjacent edges are uniformly bounded away from 0
3. $\forall e, f$ adjacent edges, $\frac{1}{M} \leq \frac{\text{diam}(e)}{\text{diam}(f)} \leq M$
4. $\forall e, f$ non-adjacent edges, $\frac{\text{diam}(e)}{\text{dist}(e,f)} \leq M$

then $\exists r_0 > 0$ such that

$\forall \Omega \text{ c.c. of } \mathbb{C} \setminus T, \forall \text{ square } Q \subset \mathbb{H}_r \text{ that has a } \tau|_\Omega \text{-edge as one side,}$

$Q \subset \tau|_\Omega \left( T(r_0) \cap \Omega \right)$
Lemma 0

If $T$ has bounded geometry, then $\exists r_0 > 0$ such that

$$\forall \Omega \text{ c.c. of } \mathbb{C}\backslash T, \forall \text{ square } Q \subset \mathbb{H}_r \text{ that has a } \tau|_\Omega\text{-edge as one side, } Q \subset \tau|_\Omega \left( T(r_0) \cap \Omega \right)$$
Theorem 1 (Bishop 2012)

If \((T, \tau)\) satisfies the following conditions

1. \(T\) has bounded geometry
2. every edge has \(\tau\)-size \(\geq \pi\)

then \(\exists\) an entire function \(f\) and a quasiconformal map \(\phi\) such that

\[ f \circ \phi = \cosh \circ \tau \text{ off } T(r_0) \]

Moreover

- \(f\) has exactly two critical values, \(-1\) and \(+1\)
- \(f\) has no finite asymptotic values
- \(\phi(T) \subset f^{-1}([-1, +1]) \) \((= \phi(T'))\)
- \(\forall c\) critical point of \(f\), \(\deg_{loc}(c, f) = \deg(c, \phi(T'))\)
Generalization: Can we construct $f$ with
- more critical values than only $-1$ and $+1$?
- some finite asymptotic values?
- arbitrary high degree critical points?
Solution: Let $T$ be an infinite bipartite graph.
**Solution:** Let $T$ be an infinite bipartite graph.

The c.c. of $\mathbb{C} \setminus T$ are sorted in three different types:

- **R-components:** $\tau|_{\Omega}: \Omega \rightarrow \mathbb{H}_r$ conformally
- **D-components:** $\tau|_{\Omega}: \Omega \rightarrow \mathbb{D}$ conformally
- **L-components:** $\tau|_{\Omega}: \Omega \rightarrow \mathbb{H}_\ell$ conformally

where $\rho_{\Omega}: D \rightarrow D$ is quasiconformal with $\rho_{\Omega}(z) = z$, $\forall z \in \partial D$. 
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|---|---|---|---|---|---|
| **R** |   |   |   |   |   |
| **D** | \((\Omega, \star)\) | $\tau|_{\Omega}$ | \((\mathbb{D}, 0)\) |   |   |
| **L** | \( \Omega \) | $\tau|_{\Omega}$ | $\mathbb{H}_\ell$ |   |   |
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More precisely:

|   | $\Omega$ | $\tau|_{\Omega}$ | $\mathbb{H}_r$ | $\cosh$ | $\mathbb{C}\setminus[-1, +1]$ |
|---|---------|-----------------|--------------|--------|-------------------------------|
| **R** |         |                 |              |        |                               |
| **D** | $(\Omega, \star)$ | $\tau|_{\Omega}$ | $(\mathbb{D}, 0)$ | $z \mapsto z^{d_\Omega}$ | $(\mathbb{D}, 0)$ |
| **L** | $(\Omega, \infty)$ | $\tau|_{\Omega}$ | $(\mathbb{H}_\ell, -\infty)$ | $\exp$ | $(\mathbb{D}, 0)$ |
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| L  | $(\Omega, \infty)$ | $\tau|_{\Omega}$ | $(\mathbb{H}_\ell, -\infty)$ | $\exp$ | $(\mathbb{D}, 0)$ | $\rho_{\Omega}$ | $(\mathbb{D}, v_{\Omega})$ |

where $\rho_{\Omega} : \mathbb{D} \to \mathbb{D}$ is quasiconformal with $\rho_{\Omega}(z) = z$, $\forall z \in \partial \mathbb{D}$. 
Theorem 2 (Bishop 2012)

If \((T, \tau)\) satisfies the following conditions

1. \(T\) has bounded geometry
2. on \(R\)-components, every edge has \(\tau\)-size \(\geq \pi\)
3. \(D, L\)-components only share edges with \(R\)-components

then \(\forall (d_\Omega \geq 2, w_\Omega \in \frac{3}{4} \mathbb{D})_{\Omega \in \{D\text{-components}\}}\) and \((v_\Omega \in \frac{3}{4} \mathbb{D})_{\Omega \in \{L\text{-components}\}}\), 

\exists an entire function \(f\) and a quasiconformal map \(\phi\) such that

\[
f \circ \phi = \sigma \circ \tau \text{ off } T(r_0) \text{ with } \sigma(z) = \begin{cases} 
\cosh(z) & \text{on } R\text{-components} \\
\rho_\Omega(z^{d_\Omega}) & \text{on } D\text{-components} \\
\rho_\Omega(\exp(z)) & \text{on } L\text{-components}
\end{cases}
\]

Moreover

- quasiconformal foldings only occur in \(R\)-components
- the only critical values of \(f\) are \(\pm 1\) and \((w_\Omega)_{\Omega \in \{D\text{-components}\}}\)
- the only asymptotic values of \(f\) are \((v_\Omega)_{\Omega \in \{L\text{-components}\}}\)
- \(\forall D\text{-component } \Omega, \exists c \in \phi(\Omega) \text{ crit. point of } f \text{ with } \deg_{\text{loc}}(c, f) = d_\Omega\)
Corollary (Bishop 2012)

Let $E, F \subset \mathbb{C}$ be two bounded countable sets with $\text{card}(E) \geq 2$. Then there exists an entire function $f$ such that

- $E$ is the set of critical values of $f$
- $F$ is the set of asymptotic values of $f$
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Bishop’s construction 2: Sketch of the proof by quasiconformal foldings

Sébastien Godillon
Theorem 1 (Bishop 2012)

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then \(\exists\) an entire function \(f\) and a quasiconformal map \(\phi\) such that

\[ f \circ \phi = \cosh \circ \tau \text{ off } T(r_0) \]

Moreover

- \(f\) has exactly two critical values, \(-1\) and \(+1\)
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- \(\phi(T) \subset f^{-1}([-1, +1]) \quad (= \phi(T'))\)
- \(\forall c\) critical point of \(f\), \(\deg_{\text{loc}}(c, f) = \deg(c, \phi(T'))\)
Idea of the proof: Construct \((T', \eta)\) such that

- \(T \subset T' \subset T(r_0)\)
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Main problem: the behavior of \( \cosh \) on the two \( \tau \)-edges of \( e \), \( \forall \) edge \( e \).
Idea of the proof: Construct \((T', \eta)\) such that
\[ T \subset T' \subset T(r_0) \]
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\[ \cosh \circ \eta \text{ continuously extends across } T' \]

Main problem: the behavior of \(\cosh\) on the two \(\tau\)-edges of \(e\), \(\forall\) edge \(e\).

More precisely, \(\forall n \in \mathbb{Z}, \cosh : i\pi[n, n + 1] \xrightarrow{\text{homeo}} [-1, +1]\)

but

1. the two \(\tau\)-edges of \(e\) are not of the form \(i\pi[n, n + 1]\) in general
2. the two \(\tau\)-edges of \(e\) have different size in general (but \(\geq \pi\))
Particular case: \( \forall \) edge \( e \), the two \( \tau \)-edges of \( e \) have same size \( \geq \pi \).

**Lemma 1**

\( \exists K \geq 1 \) such that

\( \forall \Omega \) c.c. of \( \mathbb{C} \setminus T \), \( \exists \) a map \( (\lambda_{\Omega} \circ \iota_{\Omega}) : \tau|_{\Omega}(\Omega) = \mathbb{H}_r \to \mathbb{H}_r \) such that

(i) \( (\lambda_{\Omega} \circ \iota_{\Omega}) = \text{Id off} \tau|_{\Omega}\left(T(r_0) \cap \Omega\right) \)

(ii) \( (\lambda_{\Omega} \circ \iota_{\Omega}) : \tau|_{\Omega}(\Omega) = \mathbb{H}_r \to \mathbb{H}_r \) is \( K \)-quasiconformal

(iii) \( \forall \) edge \( e \subset \partial \Omega_1 \cap \partial \Omega_2 \), \( (\lambda_{\Omega_j} \circ \iota_{\Omega_j}) \circ \tau|_{\Omega_j} \) continuously extends to \( e \) with

\[
\begin{cases}

\left((\lambda_{\Omega_j} \circ \iota_{\Omega_j}) \circ \tau|_{\Omega_j}\right)(e) = i\pi[n_j, n_j + (2k + 1)] & \text{with } n_j \in \mathbb{Z}, k \in \mathbb{N} \\
(\lambda_{\Omega_1} \circ \iota_{\Omega_1}) \circ \tau|_{\Omega_1} - (\lambda_{\Omega_2} \circ \iota_{\Omega_2}) \circ \tau|_{\Omega_2} = i\pi(n_1 - n_2) \in i\pi2\mathbb{Z} & \text{on } e
\end{cases}
\]
\[
\left\{ \begin{array}{l}
\iota_\Omega : \mathbb{H}_r \to \mathbb{H}_r \text{ moves the vertices into } i\pi\mathbb{Z} \\
\lambda_\Omega : \mathbb{H}_r \to \mathbb{H}_r \text{ fixes } i\pi\mathbb{Z} \text{ and makes the continuity across } T
\end{array} \right.
\]
\[
\begin{align*}
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\lambda: & \mathbb{H}_r \to \mathbb{H}_r \text{ fixes } i\pi\mathbb{Z} \text{ and makes the continuity across } T
\end{align*}
\]

\[
\cosh(i\pi\mathbb{Z}) = \{-1, +1\} \text{ leads to extra vertices.}
\]
Particular case: $\forall$ edge $e$, the two $\tau$-edges of $e$ have same size $\geq \pi$.

Using Lemma 1, define

$$
\begin{cases}
\eta_{|\Omega} = (\lambda_{\Omega} \circ \iota_{\Omega}) \circ \tau_{|\Omega}, \ \forall \Omega \text{ c.c. of } \mathbb{C}\setminus T \\
T' = T \text{ with extra vertices coming from } \eta^{-1}(i\pi\mathbb{Z})
\end{cases}
$$

then

(i) $\implies$ $\eta = \tau$ off $T(r_0)$
(ii) $\implies$ $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is $K$-quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C}\setminus T'$
(iii) $\implies$ $\cosh \circ \eta$ continuously extends across $T'$
General case: by proceeding as for the particular case, we may assume that

\[ \forall \text{ edge } e \subset \partial \Omega_1 \cap \partial \Omega_2, \quad \tau|_{\Omega_j} \text{ continuously extends to } e \text{ with } \]
\[ \tau|_{\Omega_j}(e) = i\pi [n_j, n_j + (2k_j + 1)] \quad \text{with } n_j \in \mathbb{Z}, k_j \in \mathbb{N} \]
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**Lemma 2 (quasiconformal folding)**

\( \exists K \geq 1 \) such that
\( \forall \Omega \text{ c.c. of } \mathbb{C} \setminus T, \exists \) a map \( \psi_{\Omega} : W_{\Omega} \subset \mathbb{H}_r \rightarrow \mathbb{H}_r \) such that

(i) \( \partial W_{\Omega} \text{ is a smooth tree with } \partial \mathbb{H}_r \subset \partial W_{\Omega} \subset \tau|_{\Omega}\left(T(r_0) \cap \Omega\right) \)

(ii) \( \psi_{\Omega} : W_{\Omega} \subset \mathbb{H}_r \rightarrow \mathbb{H}_r \) is \( K \)-quasiconformal

(iii) \( \forall \text{ edge } e \subset \partial W_{\Omega_1} \cap \partial W_{\Omega_2}, \psi_{\Omega_j} \circ \tau|_{\Omega_j} \text{ continuously extends to } e \text{ with } \)

\[
\begin{cases}
\left(\psi_{\Omega_j} \circ \tau|_{\Omega_j}\right)(e) = i\pi[m_j, m_j + 1] & \text{with } m_j \in \mathbb{Z} \\
\psi_{\Omega_1} \circ \tau|_{\Omega_1} - \psi_{\Omega_2} \circ \tau|_{\Omega_2} = i\pi(m_1 - m_2) \in i\pi2\mathbb{Z} & \text{on } e
\end{cases}
\]
\( \psi_\Omega : W_\Omega \to \mathbb{H}_r \) maps every \( \tau \)-edge onto a segment in \( \partial \mathbb{H}_r \) of length \( \pi \)
Problem: Find a quasiconformal map $\psi$ from a square to itself such that

\[
\begin{align*}
\psi & \text{ maps the left side to an edge of length } \pi \\
\psi & \text{ acts as identity on the right side}
\end{align*}
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Problem: Find a quasiconformal map $\psi$ from a square to itself such that

\[
\begin{aligned}
\psi & \text{ maps the left side to an edge of length } \pi \\
\psi & \text{ acts as identity on the right side}
\end{aligned}
\]

Solution: Add some extra edges and “unfold”.

$\psi^{-1}$ is called a quasiconformal folding.
BUT

the dilatation of $\psi$ should be uniformly bounded independently of the square size.
ψ^{-1}
$\psi_\Omega : W_\Omega \to \mathbb{H}_r$ maps every $\tau$-edge onto a segment in $\partial \mathbb{H}_r$ of length $\pi$
$\psi_{\Omega} : W_\Omega \to \mathbb{H}_r$ maps every $\tau$-edge onto a segment in $\partial \mathbb{H}_r$ of length $\pi$

$\tau^{-1}_{|\Omega}(\partial W_\Omega)$ leads to extra vertices and edges.
General case: by proceeding as for the particular case, we may assume that

\[ \forall \text{ edge } e \subset \partial \Omega_1 \cap \partial \Omega_2, \quad \tau_{|\Omega_j} \text{ continuously extends to } e \text{ with } \]

\[ \tau_{|\Omega_j}(e) = i\pi[n_j, n_j + (2k_j + 1)] \quad \text{with } n_j \in \mathbb{Z}, k_j \in \mathbb{N} \]

Lemma 2 (quasiconformal folding)

\[ \exists K \geq 1 \text{ such that } \]

\[ \forall \Omega \text{ c.c. of } \mathbb{C} \setminus T, \exists \text{ a map } \psi_{\Omega} : W_{\Omega} \subset \mathbb{H}_r \rightarrow \mathbb{H}_r \text{ such that } \]

\[ (o) \quad \partial W_{\Omega} \text{ is a smooth tree with } \partial \mathbb{H}_r \subset \partial W_{\Omega} \subset \tau_{|\Omega} (T(r_0) \cap \Omega) \]

\[ (i) \quad \psi_{\Omega} = \text{Id off } \tau_{|\Omega} (T(r_0) \cap \Omega) \]

\[ (ii) \quad \psi_{\Omega} : W_{\Omega} \subset \mathbb{H}_r \rightarrow \mathbb{H}_r \text{ is } K\text{-quasiconformal} \]

\[ (iii) \quad \forall \text{ edge } e \subset \partial W_{\Omega_1} \cap \partial W_{\Omega_2}, \psi_{\Omega_j} \circ \tau_{|\Omega_j} \text{ continuously extends to } e \text{ with } \]

\[ \begin{cases} 
    (\psi_{\Omega_j} \circ \tau_{|\Omega_j})(e) = i\pi[m_j, m_j + 1] & \text{with } m_j \in \mathbb{Z} \\
    \psi_{\Omega_1} \circ \tau_{|\Omega_1} - \psi_{\Omega_2} \circ \tau_{|\Omega_2} = i\pi(m_1 - m_2) \in i\pi2\mathbb{Z} & \text{on } e
\end{cases} \]
General case: by proceeding as for the particular case, we may assume that

\[ \forall \text{ edge } e \subset \partial \Omega_1 \cap \partial \Omega_2, \quad \tau|_{\Omega_j} \text{ continuously extends to } e \text{ \text{ with}} \]

\[ \tau|_{\Omega_j}(e) = i\pi[n_j, n_j + (2k_j + 1)] \quad \text{ with } n_j \in \mathbb{Z}, k_j \in \mathbb{N} \]

Using Lemma 2, define

\[
\begin{cases}
\eta|_{\Omega} = \psi_{\Omega} \circ \tau|_{\Omega}, \quad \forall \Omega \text{ c.c. of } \mathbb{C} \setminus T \\
T' = T \text{ with extra vertices and edges coming from } \eta^{-1}(\partial \mathbb{H}_r)
\end{cases}
\]

Then

(i) \[ \implies \eta = \tau \text{ off } T(r_0) \]

(ii) \[ \implies \eta|_{\Omega'} : \Omega' \to \mathbb{H}_r \text{ is } K\text{-quasiconformal, } \forall \Omega' \text{ c.c. of } \mathbb{C} \setminus T' \]

(iii) \[ \implies \cosh \circ \eta \text{ continuously extends across } T' \]
Tak for din opmærksomhed!
Wandering under Bishop’s trees

Existence and non-existence of wandering domains for entire functions

Sébastien Godillon
Wandering domain

Let \( f \) be a rational map or a transcendental entire function. A Fatou domain \( U \) of \( f \) is said to be wandering if

\[
\forall n \neq m, \quad f^n(U) \cap f^m(U) = \emptyset
\]

Sullivan, 1985

If \( f \) is a rational map then \( f \) has no wandering domains.

Main tools: quasiconformal deformations

Singular set

Let \( f : \mathbb{C} \to \mathbb{C} \) be a transcendental entire function. Denote by \( S(f) = \text{Crit}(f) \cup \text{Asym}(f) \) the set of finite singular values. Eremenko-Lyubich, Goldberg-Keen, 1986

If \( |S(f)| < +\infty \) then \( f \) has no wandering domains.

Main tools: quasiconformal deformations
Wandering domain

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Eremenko-Lyubich, Goldberg-Keen, 1986

If \( |S(f)| < +\infty \) then \( f \) has no wandering domains.

Main tools: quasiconformal deformations
Baker, 1975

If $U$ is a multiply connected Fatou domain of $f$ then $U$ is wandering.

Baker, 1976

$$g(z) = \frac{1}{4e} z^2 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{\gamma_n} \right)$$

for suitable $\gamma_n > 1$

has (multiply connected and hence) wandering domains.
Baker, 1976

\[ g(z) = \frac{1}{4e} z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{\gamma_n}\right) \text{ for suitable } \gamma_n > 1 \]

has (multiply connected) wandering domains.

Herman, 1981

\[
\begin{aligned}
    f_1(z) &= z - 1 + e^{-z} + 2\pi i \\
    f_2(z) &= z + \frac{e^{2\pi i \alpha} - 1}{2\pi} \sin(2\pi z) + 1
\end{aligned}
\]

for suitable \( \alpha \in \mathbb{R} \)

both have (simply connected) wandering domains.
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\quad \text{for suitable } \alpha \in \mathbb{R}
\]

both have (simply connected) wandering domains.

(Devaney et al., 1989) ??

\[ f_3(z) = z + 2\pi \sin(z) \]

has wandering domains.
Fatou, 1920

If $U$ is wandering then every limit function of $\{f^n|U\}_{n \geq 1}$ is constant. In particular, $U$ is either:

- **escaping:** $\forall(n_k)$, $f^{n_k}|U \xrightarrow[k \to +\infty]{} \infty$

- **oscillating:** $\exists(n_k, m_k)$, $f^{n_k}|U \xrightarrow[k \to +\infty]{} \infty$ and $f^{m_k}|U \xrightarrow[k \to +\infty]{} a \in J(f)$

- **bounded:** $\forall(n_k)$, $f^{n_k}|U \not\xrightarrow[k \to +\infty]{} \infty$
Fatou, 1920

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- **oscillating**: $\exists (n_k, m_k), \quad f^{n_k}|_U \xrightarrow[k \to +\infty]{} \infty$ and $f^{m_k}|_U \xrightarrow[k \to +\infty]{} a \in \mathcal{J}(f)$
- **bounded**: $\forall (n_k), \quad f^{n_k}|_U \xrightarrow[k \to +\infty]{} \infty$

Baker, 1976

If $f^{n_k}|_U \xrightarrow[k \to +\infty]{} a$ then $a \in \overline{E} \cup \{\infty\}$ where $E = \bigcup_{s \in S(f)} \bigcup_{n \geq 1} f^n(s)$.
Fatou, 1920

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Bergweiler et al., 1993

If $f^{n_k}|_U \xrightarrow[k \to +\infty]{} a$ then $a \in E' \cup \{\infty\}$.

$z \mapsto \exp(z), \quad z \mapsto \frac{\sin(z)}{z}, \quad$ and $z \mapsto \frac{\pi^2}{\pi^2 - z^2} \sin(z)$ have no wandering domains.
∃ an entire function $f$ which has (Fatou domains with infinitely many finite constant limit functions and hence) oscillating wandering domains.

Main tool: approximation theory
Eremenko-Lyubich, 1987

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Main tool: approximation theory

Singh, 2003

∃ two entire functions $f, g$ and a domain $U \subset \mathbb{C}$ which lies in

\[
\begin{cases}
\text{a periodic domain} & \text{for } f, g, \text{ and } g \circ f \\
\text{a wandering domain} & \text{for } f \circ g
\end{cases}
\]

However $g \circ f$ must have wandering domains (Bergweiler-Wang, 1998).

Main tool: approximation theory
Eremenko-Lyubich, 1987

\[ \exists \text{ an entire function } f \text{ which has (Fatou domains with infinitely many finite constant limit functions and hence) oscillating wandering domains.} \]

**Main tool:** approximation theory

Singh, 2003

\[ \exists \text{ two entire functions } f, g \text{ and a domain } U \subset \mathbb{C} \text{ which lies in} \]

\[ \begin{align*}
\{ & \text{a periodic domain for } f, g, \text{ and } g \circ f \\
& \text{a wandering domain for } f \circ g
\end{align*} \]

However, \( g \circ f \) must have wandering domains (Bergweiler-Wang, 1998).

**Main tool:** approximation theory

**Question:** Do there exist \( f, g \) which both have no wandering domains but whose composition \( f \circ g \) has some?
Eremenko-Lyubich’s class

\[ \mathcal{B} = \left\{ f : \mathbb{C} \to \mathbb{C} \text{ entire function such that } S(f) \text{ is bounded} \right\} \]
Eremenko-Lyubich’s class

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Eremenko-Lyubich, 1992

If \( f \in \mathcal{B} \) then any wandering domain is either oscillating or bounded.

Main result: \( \mathcal{I}(f) \subset \mathcal{J}(f) \) (and hence \( \mathcal{J}(f) = \overline{\mathcal{I}(f)} \))
**Eremenko-Lyubich’s class**

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If \( f \in B \) then any wandering domain is either oscillating or bounded.

**Main result:** \( \mathcal{I}(f) \subset \mathcal{J}(f) \) (and hence \( \mathcal{J}(f) = \overline{\mathcal{I}(f)} \))

**Mihaljević-Rempe, 2012**

If \( f \in B \) satisfies \( \sup_{s \in S(f)} |f^n(s)| \xrightarrow{n \to +\infty} +\infty \) and a certain condition (⋆) then \( f \) has no wandering domains.

**Main tool:** hyperbolic geometry

\[ z \mapsto \lambda \frac{\sinh(z)}{z} + a \] for every \( \lambda, a \in \mathbb{R} \) has no wandering domains.
∃ an entire function $f \in \mathcal{B}$ which has (oscillating) wandering domains (with infinitely many finite constant limit functions).

Main tool: Bishop’s construction by quasiconformal foldings
Wandering under Bishop’s trees
-
Examples of wandering domains in Eremenko-Lyubich’s class

Sébastien Godillon
∃ an entire function $f \in \mathcal{B}$ which has (oscillating) wandering domains (with infinitely many finite constant limit functions).

Main tool: Bishop’s construction by quasiconformal foldings
Bishop’s example is of the form:

\[ f = F \circ \phi \quad \text{with} \quad \begin{cases}  
F : \mathbb{C} \to \mathbb{C} \text{ quasiregular (transcendental)} \\
\phi : \mathbb{C} \to \mathbb{C} \text{ quasiconformal so that } \mu_{\phi^{-1}} = F^*(\mu_0) 
\end{cases} \]
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Moreover,

- \( \forall z \in \mathbb{C}, \ F(-z) = F(z) \) and \( F(\bar{z}) = \overline{F(z)} \)
- \( \text{Crit}(F) = \{-1, +1\} \cup \{w_n, \ n \geq 1\} \cup \{\frac{1}{2}\} \subset \overline{\mathbb{D}} \) with \( w_n \xrightarrow{n \to +\infty} \frac{1}{2} \)
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- \( \text{Asym}(F) = \emptyset \)

\( \implies \) the same holds for \( f \) as well.

- \( \text{supp}(F^*(\mu_0)) \) is small enough in order that we may find \( \phi \approx \text{Id}_\mathbb{C} \)
$F$ is constructed following an infinite graph.
$F$ is constructed following an infinite graph.
$F$ is constructed following an infinite graph.
$F$ maps \{ straight lines \ onto \ $[-1, +1]$ \\
circle arcs \ onto \ $\partial \mathbb{D}$ \}
$F : S^+ \xrightarrow{\lambda \sinh} \mathbb{H}_r \xrightarrow{\cosh} \mathbb{C} \setminus [-1, +1]$

for some parameter $\lambda > 0$. 

\[ z \quad \lambda \sinh \quad \mathbb{H}_r \quad \cosh \quad \mathbb{C} \setminus [-1, +1] \cosh(\lambda \sinh(z)) \]
for some parameter $\lambda > 0$. 
For every $n \geq 1$, 

$$F : (D_n, z_n) \xrightarrow{z \mapsto (z - z_n)^{d_n}} (\mathbb{D}, 0) \xrightarrow{\rho_n} (\mathbb{D}, w_n)$$

with 

$$\left\{ \begin{array}{l}
\rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal} \\
\rho_n(0) = w_n
\end{array} \right.$$ 

for some parameters $d_n \xrightarrow{n \to +\infty} +\infty$ and $w_n \xrightarrow{n \to \infty} \frac{1}{2}$. 
For every \( n \geq 1 \),

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F : (D_n, z_n) \xrightarrow{z \mapsto (z - z_n)^{d_n}} (\mathbb{D}, 0) \xrightarrow{\rho_n} (\mathbb{D}, w_n)
\]

with

\[
\rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal}
\]

\[
\rho_n(0) = w_n
\]

\[
\text{supp}(\mu_{\rho_n}) \subset \{ \frac{1}{2} \leq |z| \leq 1 \}
\]

for some parameters \( d_n \xrightarrow{n \to +\infty} +\infty \) and \( w_n \xrightarrow{n \to \infty} \frac{1}{2} \).
Using Bishop’s construction by quasiconformal foldings, $F$ may be extended to a quasiregular map $F : \mathbb{C} \to \mathbb{C}$ such that:

- $\forall z \in \mathbb{C}, F(-z) = F(z)$ and $F(\overline{z}) = \overline{F(z)}$
- $\text{Crit}(F) = \{-1, +1\} \cup \{w_n, \ n \geq 1\} \cup \{\frac{1}{2}\} \subset \overline{D}$ with $w_n \xrightarrow{n \to +\infty} \frac{1}{2}$
- $\text{Asym}(F) = \emptyset$

Let $f = F \circ \phi$ with $\phi : \mathbb{C} \to \mathbb{C}$ quasiconformal so that $\mu_{\phi^{-1}} = F^*(\mu_0)$. 

![Diagram showing the concept of Bishop's construction with quasiconformal foldings and the extension of $F$ to a quasiregular map.](image-url)
Using Bishop’s construction by quasiconformal foldings, $F$ may be extended to a quasiregular map $F : \mathbb{C} \to \mathbb{C}$ such that:

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Using Bishop’s construction by quasiconformal foldings, $F$ may be extended to a quasiregular map $F : \mathbb{C} \to \mathbb{C}$ such that:

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Let $f = F \circ \phi$ with $\phi : \mathbb{C} \to \mathbb{C}$ quasiconformal so that $\mu_{\phi^{-1}} = F^*(\mu_0)$. 
Choice of the parameters \((\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})\)

- \(\lambda > 0\) is fixed so that \(f^n \left( \frac{1}{2} \right) \xrightarrow[n \to +\infty]{} +\infty\) very fast.

\[\forall x \in \mathbb{R}, \quad f(x) = \cosh \left( \lambda \sinh \left( \varphi \big| \mathbb{R} \big( x \big) \right) \right) \approx \frac{1}{2} \exp \left( \frac{\lambda}{2} \exp(x) \right)\]
\[ S^+ \]

\[ 1/2 \] \[ f^{(1/2)} \] \[ f^{n-1(1/2)} \] \[ f^n(1/2) \] \[ f^{n+1(1/2)} \]

\[ \tilde{D}_n \]

\[ \tilde{D}_{n+1} \]
\[ S^+ \quad f(1/2) \quad f^{n-1}(1/2) \quad f^n(1/2) \quad f^{n+1}(1/2) \]

\[ D_n \]

\[ \tilde{D}_n \]

\[ \tilde{D}_{n+1} \]
\[ f_{n+1}(1/2) = \frac{1}{4} \tilde{D}_n \]

and inradius \((U_{n+1}) \geq C\) and

\[ (df_n dx)_{\mid x = 1/2} - 1 f_n + 1 (U_{n+1}) = \frac{1}{4} \tilde{D}_n + 1 \]

and inradius \((U_{n+1}) \geq C\) and diam \((f(U_{n+1})) \leq C'\).
\[ f^n(U_n) = \frac{1}{4} \tilde{D}_n \quad \text{and} \quad \text{inradius}(U_n) \geq C \left( \left. \frac{df^n}{dx} \right|_{x=\frac{1}{2}} \right)^{-1} \]
\[ f^n(U_n) = \frac{1}{4} \tilde{D}_n \quad \text{and} \quad \text{inradius}(U_n) \geq C. \left( \frac{df^n}{dx} \bigg|_{x=\frac{1}{2}} \right)^{-1} \]

\[ f^{n+1}(U_{n+1}) = \frac{1}{4} \tilde{D}_{n+1} \quad \text{and} \quad \text{inradius}(U_{n+1}) \geq C. \left( \frac{df^{n+1}}{dx} \bigg|_{x=\frac{1}{2}} \right)^{-1} \]
\[ f^{n+1}(U_{n+1}) = \frac{1}{4} \tilde{D}_{n+1} \quad \text{and} \quad \text{inradius}(U_{n+1}) \geq C \left( \frac{d f^{n+1}}{d x} \bigg|_{x=\frac{1}{2}} \right)^{-1} \]

\[ \tilde{w}_n \in f \left( \frac{1}{4} \tilde{D}_n \right) \quad \text{and} \quad \text{diam} \left( f \left( \frac{1}{4} \tilde{D}_n \right) \right) \leq C' \left( \frac{1}{4} \tilde{d}_n \right) \]
Choice of the parameters \((\lambda, (d_n)_{n \geq 1}, (w_n)_{n \geq 1})\)

- \(\lambda > 0\) is fixed so that \(f^n \left( \frac{1}{2} \right) \xrightarrow[n \to +\infty]{} +\infty\) very fast.

- \(\tilde{d}_n \xrightarrow[n \to +\infty]{} +\infty\) and \(\tilde{w}_n \xrightarrow[n \to \infty]{} \frac{1}{2}\) are fixed so that

\[
\forall n \geq N, \quad f^{n+1}(U_n) = f \left( \frac{1}{4} \tilde{D}_n \right) \subset U_{n+1}.
\]
Choice of the parameters \((\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})\)

- \(\lambda > 0\) is fixed so that \(f^n \left(\frac{1}{2}\right) \xrightarrow[n\to\infty]{} +\infty\) very fast.

- \(\tilde{d}_n \xrightarrow[n\to\infty]{} +\infty\) and \(\tilde{w}_n \xrightarrow[n\to\infty]{} \frac{1}{2}\) are fixed so that

\[
\forall n \geq N, \quad f^{n+1}(U_n) = f \left(\frac{1}{4} \tilde{D}_n\right) \subset U_{n+1}.
\]

Therefore,

\[
U_N \xrightarrow{f^{N+1}} U_{N+1} \xrightarrow{f^{N+2}} U_{N+2} \xrightarrow{f^{N+3}} U_{N+3} \xrightarrow{f^{N+4}} \ldots
\]
Bishop’s example has no unexpected wandering domains.
Main ingredients of the proof:
Let $W$ be a wandering domain of $f$ (in the upper half plane).

1. **Baker’s argument**

2. **Mihaljević-Rempe’s hyperbolic geometry lemma**
Main ingredients of the proof:
Let $W$ be a wandering domain of $f$ (in the upper half plane).

1. **Baker’s argument** (here $E' = \{ f^n \left( \frac{1}{2} \right) \}_{n \geq 1}$)

   \[ \exists (n_k) \text{ such that } \begin{cases} f^{n_k}|_{W} \xrightarrow{k \to +\infty} \frac{1}{2} \\ f^{n_k-1}(W) \subset D_{m_k} \text{ for some } m_k \end{cases} \]

2. **Mihaljević-Rempe’s hyperbolic geometry lemma**
Main ingredients of the proof:

Let $W$ be a wandering domain of $f$ (in the upper half plane).

1. **Baker’s argument** (here $E' = \{f^n \left(\frac{1}{2}\right)\}_{n\geq 1}$)

   $$\exists (n_k) \text{ such that } \begin{cases} f^{n_k}|_W \xrightarrow[k\to+\infty]{} \frac{1}{2} \\ f^{n_k-1}(W) \subset D_{m_k} \text{ for some } m_k \end{cases}$$

2. **Mihaljević-Rempe’s hyperbolic geometry lemma**

   $$\text{dist}_U \left(f^{n_k-1}(W), U\setminus f^{-1}(\mathbb{D})\right) \xrightarrow[k\to+\infty]{} +\infty$$

   where $$U = \mathbb{C}\setminus \left(\left[\frac{1}{2}, +\infty\right]\cup \bigcup_{n\geq N} \bigcup_{j=1}^n f^j(U_n)\right)$$
Bishop’s example has no unexpected wandering domains.

Main ingredients of the proof:

Let \( W \) be a wandering domain of \( f \) (in the upper half plane).

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   \]

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   \[ \implies \begin{cases} 
   f^{n_k-1}(W) \subset \tilde{D}_{p_k} \text{ for some } p_k \\
   \text{dist}_\mathbb{C} \left( f^{n_k-1}(W), \frac{1}{4} \tilde{D}_{p_k} \right) \xrightarrow{k \to +\infty} 0
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Main ingredients of the proof:

Let $W$ be a wandering domain of $f$ (in the upper half plane).

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Therefore $W$ is eventually mapped into some $U_n$. 

■
Bishop’s example has no unexpected wandering domains.

Corollary

∃ two entire functions $f, g$ (in $B$) which both have no wandering domains but whose composition $f \circ g$ has some.
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**Corollary**

∃ two entire functions \( f, g \) (in \( B \)) which both have no wandering domains but whose composition \( f \circ g \) has some.

**Strategy of the proof:**

Construct \( f, g \) like Bishop’s example.

Chose the parameters \( \tilde{d}_n \xrightarrow{n \to +\infty} +\infty \) and \( \tilde{w}_n \xrightarrow{n \to \infty} \frac{1}{2} \) so that:

\[
\forall k \geq N, \quad \begin{cases}
  f^{4k+1}(U_{4k}) & \subset U_{4k+1} \\
  f^{4k+2}(U_{4k+1}) & \subset U_{4k+2}
\end{cases}
\quad \text{and} \quad \begin{cases}
  g^{4k+3}(U_{4k+2}) & \subset U_{4k+3} \\
  g^{4k+4}(U_{4k+3}) & \subset U_{4k+4}
\end{cases}
\]
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**Corollary**

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**Strategy of the proof:**

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\begin{align*}
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\end{align*}
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and

\[
\begin{align*}
 g^{4k+1}(U_{4k}) & \subset U_{4k+1} \\
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\end{align*}
\]