From a tree to a Persian carpet

Sébastien Godillon
Definition

\( R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) rational map of degree \( d \geq 2 \)

- The Julia set:
  \[ \mathcal{J}(R) = \{ \text{repelling periodic points} \} \]

- The Fatou set:
  \[ \mathcal{F}(R) = \hat{\mathbb{C}} - \mathcal{J}(R) = \{ \text{points with stable behavior} \} \]
Definition

$R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational map of degree $d \geq 2$

- The Julia set:
  $$\mathcal{J}(R) = \{\text{repelling periodic points}\}$$

- The Fatou set:
  $$\mathcal{F}(R) = \hat{\mathbb{C}} - \mathcal{J}(R) = \{\text{points with stable behavior}\}$$

Proposition

$\mathcal{J}(R)$ is a fully invariant non-empty perfect compact set. Furthermore

- either $\mathcal{J}(R)$ is connected,
- or else $\mathcal{J}(R)$ has uncountably many connected components.
Definition

A point $z \in \mathcal{J}(R)$ is said buried if

$$\forall \text{ Fatou component } F, z \notin \partial F$$
Definition

A point \( z \in \mathcal{J}(R) \) is said **buried** if

\[
\forall \text{ Fatou component } F, \ z \notin \partial F
\]
Definition

A point $z \in \mathcal{J}(R)$ is said buried if

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**Question 0** Does there exist buried Julia components?
## Proposition

If $P$ is a polynomial then

\[ J(P) = \partial B(\infty) \]
Proposition

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Proposition

If $R$ is of degree $d = 2$ then

- either $J(R)$ is connected,
- or else $J(R)$ is a Cantor set.
Proposition

If $P$ is a polynomial then

$$\mathcal{J}(P) = \partial B(\infty)$$

Proposition

If $R$ is of degree $d = 2$ then

- either $\mathcal{J}(R)$ is connected,
- or else $\mathcal{J}(R)$ is a Cantor set.

Proposition

If the complement of every Julia component of $\mathcal{J}(R)$ is connected then there is no buried Julia component in $\mathcal{J}(R)$. 
Theorem (McMullen 91)

\[ g_{n,d,\varepsilon} : z \mapsto z^n + \frac{\varepsilon}{z^d} \]

If \(|\varepsilon| > 0\) is small enough and if

\[ \frac{1}{n} + \frac{1}{n} < 1 \] (H0)

then \(\mathcal{J}(g_{n,d,\varepsilon})\) is a Cantor of Jordan curves.
A Cantor of Jordan curves
Theorem (McMullen 91)

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In particular, every wandering Julia component is buried.
Theorem (McMullen 91)

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If \(|\varepsilon| > 0\) is small enough and if

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then \( J(g_{n,d,\varepsilon}) \) is a Cantor of Jordan curves.

In particular, every wandering Julia component is buried.

**Question 1** Does there exist buried Julia components for rational maps of degree \( d < 5 \)?
Theorem (Pilgrim-Tan Lei 00)

If $R$ is geometrically finite then every wandering Julia component of $\mathcal{J}(R)$ is

- either a point,
- or a Jordan curve.
Theorem (Pilgrim-Tan Lei 00)

If $R$ is geometrically finite then every wandering Julia component of $\mathcal{J}(R)$ is

- either a point,
- or a Jordan curve.

**Question 2** Does there exist buried Julia components which are wandering points?
Theorem (Pilgrim-Tan Lei 00)

If $R$ is geometrically finite then every wandering Julia component of $\mathcal{J}(R)$ is
- either a point,
- or a Jordan curve.

**Question 2** Does there exist buried Julia components which are wandering points?

**Question 3** Does there exist buried Julia components which are neither a point nor a Jordan curve?
Theorem

\[ f_\varepsilon : z \mapsto \frac{(1 - \varepsilon) \left[ (1 - 4\varepsilon + 6\varepsilon^2 - \varepsilon^3)z - 2\varepsilon^3 \right]}{(z - 1)^2 \left[ (1 - \varepsilon - \varepsilon^2)z - 2\varepsilon^2(1 - \varepsilon) \right]} \]

If |\( \varepsilon \)| > 0 is small enough then \( \mathcal{J}(f_\varepsilon) \) contains buried Julia components of several types:

1. wandering Jordan curves
2. wandering (and preperiodic) points
3. preperiodic Julia components which are quasiconformally homeomorphic to a finite covering space of \( \mathcal{J}(z \mapsto \frac{1}{(z-1)^2}) \)
A Persian carpet
McMullen example: $g_{n,d,\varepsilon} : z \mapsto z^n + \frac{\varepsilon}{z^d}$
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McMullen example: $g_{n,d,\varepsilon}(z) \mapsto z^n + \frac{\varepsilon}{z^d}$
Theorem

\[ \exists \text{ a continuous surjective map } \pi : \hat{\mathbb{C}} \to T \text{ such that} \]

\[ \begin{array}{ccc}
\mathcal{J}(g_n,d,\varepsilon) & \xrightarrow{g_n,d,\varepsilon} & \mathcal{J}(g_n,d,\varepsilon) \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{J}(T) & \xrightarrow{\tau} & \mathcal{J}(T)
\end{array} \]

where \( \mathcal{J}(T) \subset T \) is a Cantor set and

\[ \forall x \in \mathcal{J}(T), \pi^{-1}(x) \text{ is a Jordan curve Julia component} \]
\[ P : z \mapsto z^2 + c \text{ with } c \approx -0.157 + 1.032i \]
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Let \( \mathcal{H} \) be the associated Hubbard tree.
\((\mathcal{H}, P|_{\mathcal{H}})\) is conjugated to the dynamical tree \((T, \tau)\).
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McMullen example
From a Hubbard tree...
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$P : z \mapsto z^2 + c$ with $c \approx -0.157 + 1.032i$

Let $J(H)$ be the Cantor set $J(P) \cap H$. 
Theorem

∃ a continuous surjective map \( \pi : \hat{C} \to H \) and
∃ an invariant subset \( \mathcal{J}_H(f_\varepsilon) \subset \mathcal{J}(f_\varepsilon) \) such that

\[
\begin{align*}
\mathcal{J}_H(f_\varepsilon) & \xrightarrow{f_\varepsilon} \mathcal{J}_H(f_\varepsilon) \\
\pi & \downarrow \quad \pi \\
\mathcal{J}(H) & \xrightarrow{P|_H} \mathcal{J}(H)
\end{align*}
\]

where \( \forall x \in \mathcal{J}(H), \pi^{-1}(x) \) is a Julia component. Moreover

- \( \pi^{-1}(\alpha) \) is fixed, buried and homeomorphic to \( \mathcal{J}(z \mapsto \frac{1}{(z-1)^2}) \)
- \( \forall x \in \mathcal{J}(H) - \bigcup_{n \geq 0} (P^n)^{-1}(\alpha) \), \( \pi^{-1}(x) \) is a Jordan curve
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Proof

By quasiconformal surgery:
Proof: Branching point (Step 1)
Proof: Branching point (Step 1)

\[ c_1 = 1 \xrightarrow{2} c_2 = \infty \xrightarrow{2} c_3 = 0 \]
Proof: Branching point (Step 1)

\[ c_1 = 1 \rightarrow^2 c_2 = \infty \rightarrow^2 c_3 = 0 \]

\[ \hat{f} = (z \mapsto z^2) \circ \left( z \mapsto \frac{1}{1 - z} \right) = \left( z \mapsto \frac{1}{(1 - z)^2} \right) \]
Proof: Branching point (Step 1)
Proof: Cutting up (Step 2)
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Proof: Folding point (Step 3)

\[ c_1 = 1 \]
\[ c_0 = p \]
\[ c_3 = 0 \]
\[ c_2 = \infty \]
Proof: Folding point (Step 3)

\[ F|_A = \psi \circ \left( z \mapsto z + \frac{1}{z} \right) \circ \varphi \]
Proof: Uniformization (Step 4)
Proof: Uniformization (Step 4)

From Morrey-Ahlfors-Bers theorem,
\[ \exists \text{ a quasiconformal map } \phi \text{ such that} \]

\[
\begin{array}{ccc}
\hat{\mathcal{C}} & \xrightarrow{F} & \hat{\mathcal{C}} \\
\downarrow \phi & & \downarrow \phi \\
\hat{\mathcal{C}} & \xrightarrow{f_\epsilon} & \hat{\mathcal{C}}
\end{array}
\]
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Generalized problem:
Provide the Hubbard tree $\mathcal{H}$ with a weight function $w$. 
Provide the Hubbard tree $\mathcal{H}$ with a weight function $w$.

**Question:** Does there exist a rational map whose exchanging dynamics of Julia components is “encoded” by the weighted Hubbard tree $(\mathcal{H}, w)$?
Step 1: Branching point

\[ \text{Diagram: } c_1 \rightarrow d_1 \rightarrow \alpha \rightarrow d_0 \rightarrow c_3 \]

\[ c_2 \rightarrow d_2 \rightarrow \alpha \]

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Step 1: Branching point

\[ c_1 \xrightarrow{d_1} c_2 \xrightarrow{d_2} c_3 \]

\[ d_0 \]

Lemma 1: Topological obstruction

There exists a rational model $\hat{f}$ if and only if

\[
\hat{d} = \frac{1}{2} \left( d_0 + d_1 + d_2 - 1 \right)
\]

is an integer $\geq 2$ and

\[
\max\{d_0, d_1, d_2\} \leq \hat{d}
\]

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Step 1: Branching point

\[
\begin{align*}
  c_1 & \xrightarrow{d_1} c_2 \xrightarrow{d_2} c_3 \\
  d_0 & \downarrow
\end{align*}
\]

Lemma 1: Topological obstruction

There exists a rational model \( \hat{f} \) if and only if

\[
\begin{align*}
\hat{d} &= \frac{1}{2} (d_0 + d_1 + d_2 - 1) \text{ is an integer } \geq 2 \\
\max\{d_0, d_1, d_2\} &\leq \hat{d}
\end{align*}
\]
Step 2: Cutting up
Definition (Thurston obstruction)

\[
\begin{align*}
\tau([\alpha, c_0]) &= [\alpha, c_1] \\
\tau([\alpha, c_1]) &= [\alpha, c_2] \\
\tau([\alpha, c_2]) &= [\alpha, c_0] \cup [c_0, c_3] \\
\tau([c_0, c_3]) &= [\alpha, c_1] \cup [\alpha, c_0]
\end{align*}
\]

\[
\mathcal{H}, w \text{ is said not obstructed if } \lambda(M_{\mathcal{H}}) < 1.
\]
McMullen example

The McMullen example is not obstructed if and only if

\[ \lambda(M_T) = \frac{1}{n} + \frac{1}{d} < 1 \]
In that case: \( \lambda(M_{\mathcal{H}}) \approx 0.918 < 1 \)
Step 2: Cutting up

A weighted Hubbard tree
Some obstructions
Final statement
Step 2: Cutting up

Lemma 2: Analytical obstruction

There exists a system of equipotentials of $\hat{f}$ if and only if

$$(\mathcal{H}, w) \text{ is not obstructed} \quad (\text{H2})$$
Theorem

If the weighted Hubbard tree \((\mathcal{H}, w)\) satisfies

\[
\begin{cases}
\hat{d} = \frac{1}{2}(d_0 + d_1 + d_2 - 1) \text{ is an integer } \geq 2 \\
\max\{d_0, d_1, d_2\} \leq \hat{d}
\end{cases}
\]

(H1)

\((\mathcal{H}, w)\) is not obstructed (H2)

then \(\exists\) some rational map \(f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}\) of degree \(d = \hat{d} + d_3\) whose exchanging dynamics is “encoded” by \((\mathcal{H}, w)\).
Question: What happens for more sophisticated trees?
Thank you for your attention!