University of Cergy-Pontoise **2011-2012**
L1-MI 2nd semester

Test 1

(1 hour)

Question about course 1.
Recall all the axioms that a ring $(R, +, \times)$ must satisfy.

Question about course 2.
Recall the axioms that an automorphism φ of a group $(G, \star)$ must satisfy.

Exercise 3.
Consider the following set

$$A = \{n + k\sqrt{5} \in \mathbb{R} \,/\, n, k \in \mathbb{Z}\}$$

1. Show that $(A, +)$ is a subgroup of $(\mathbb{R}, +)$.
2. Prove that $(A, +, \times)$ is a ring.

Exercise 4.
Consider the following sets

$$F = \{f : \mathbb{R} \to \mathbb{R}\} \quad \textit{and} \quad \forall x \in \mathbb{R},\, F_x = \{f \in F \,/\, f(1) = x\}$$

1. Show that $(F_0, +)$ is a subgroup of $(F, +)$.
2. Is $(F_1, +)$ a group ? Why ?
3. More generally, find all the x in $\mathbb{R}$ for which $(F_x, +)$ is a group.
4. Prove that the map $\Phi : f \mapsto \Phi(f) = f(1)$ is a homomorphism from $(F, +)$ into $(\mathbb{R}, +)$.

Exercise 5.
Define the following sequence of integers

$$\forall n \in \mathbb{N},\, u_n = 4^{3n+2} + 8^{2n+1}$$

1. Show that $u_0 \equiv 6$ [9].
2. Show that the same holds for u_1 and u_2.
3. Let $n \in \mathbb{N}$ be a fixed integer and assume that $u_n \equiv 6$ [9]. Then prove that $u_{n+1} \equiv 6$ [9].
4. Conclude by induction.

Exercise 6.
Consider the following equation in $\mathbb{Z}/13\mathbb{Z}$

$$(E) \quad \overline{6} \times x + \overline{7} = \overline{2}$$

1. Is $(\mathbb{Z}/13\mathbb{Z} - \{\overline{0}\}, \times)$ a group ? Why ?
2. Find a congruence class $y \in \mathbb{Z}/13\mathbb{Z}$ such that $y \times \overline{6} = \overline{1}$.
3. Solve the equation (E) in $\mathbb{Z}/13\mathbb{Z}$.
4. Similarly, solve the equation (E) in $\mathbb{Z}/11\mathbb{Z}$.