Chapter 5

Matrices

In this chapter, fix a commutative field $(\mathbb{K}, +, \times)$ (for instance $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). We denote

- 0 the additive identity
- $-a$ the additive inverse of an element $a \in \mathbb{K}$
- 1 the multiplicative identity
- a^{-1} the multiplicative inverse of an element $a \in \mathbb{K}^* = \mathbb{K} - \{0\}$

5.1 Definition and operations

In this section, fix two positive integers $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$.

5.1.1 The vector space $\mathcal{M}_{p,n}(\mathbb{K})$

Definition 5.1 (*Matrix*)

*Let $(m_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$ be a family of elements in $\mathbb{K}$. The **matrix** with **entries** $(m_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$ is the following rectangular arrangement of these elements*

$$M = (m_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p,1} & m_{p,2} & \dots & m_{p,n} \end{pmatrix}$$

Moreover

- *the horizontal and vertical lines are respectively called **rows** and **columns***
- *the size $\boldsymbol{p \times n}$ of the rectangular arrangement, where p and n are respectively the numbers of rows and columns, is called the **dimension***
- *for any $i \in \{1, 2, \dots, p\}$ and any $j \in \{1, 2, \dots, n\}$, $m_{i,j}$ is called the **entry of the $\boldsymbol{i}^{th}$ row and the $\boldsymbol{j}^{th}$ column** or shortly the **$\boldsymbol{(i,j)}^{th}$ entry***

We denote $\boldsymbol{\mathcal{M}_{p,n}}(\mathbb{K})$ the set of all matrices with dimension $p \times n$ and entries in $\mathbb{K}$.

Examples :

a)

$$A = \begin{pmatrix} 1 & -1 & 0 & \sqrt{2} \\ \frac{3}{5} & 0 & 17 & 3 \\ 5\pi & 1+\sqrt{5} & 0 & -\frac{7}{2} \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}) \quad \text{or} \quad B = \begin{pmatrix} 0 & -1+\imath \\ -1-\imath & 0 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{C})$$

The $(2,3)^{\text{th}}$ entry of A is 17 and its dimension is 3×4.

b) A matrix of dimension $p \times 1$ is called a **column vector** and a matrix of dimension $1 \times n$ is called a **row vector.**

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix} \in \mathcal{M}_{p,1}(\mathbb{K}) \quad \text{and} \quad R = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix} \in \mathcal{M}_{1,n}(\mathbb{K})$$

c) A 1×1 matrix is no more than an element in $\mathbb{K}$.

Definition 5.2 (*Addition and scalar multiplication in $\mathcal{M}_{n,p}(\mathbb{K})$*)

- *Let A and B be two matrices in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries respectively $(a_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$ and $(b_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$. We define the sum of A and B, denoted $A+B$, to be the matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries $(a_{i,j}+b_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$. That provides a binary operation $+$ on $\mathcal{M}_{p,n}(\mathbb{K})$.*

$$\begin{aligned} A+B &= \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & \dots & a_{p,n} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p,1} & b_{p,2} & \dots & b_{p,n} \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \dots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \dots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,1}+b_{p,1} & a_{p,2}+b_{p,2} & \dots & a_{p,n}+b_{p,n} \end{pmatrix} \end{aligned}$$

- *Let λ be a scalar in $\mathbb{K}$ and A be a matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries $(a_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$. We define the scalar multiplication of λ with A, denoted $\lambda.A$, to be the matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries $(\lambda a_{i,j})_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant n}}$. That provides an external binary operation . over $\mathbb{K}$ on $\mathcal{M}_{p,n}(\mathbb{K})$.*

$$\lambda.A = \lambda. \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & \dots & a_{p,n} \end{pmatrix} = \begin{pmatrix} \lambda a_{1,1} & \lambda a_{1,2} & \dots & \lambda a_{1,n} \\ \lambda a_{2,1} & \lambda a_{2,2} & \dots & \lambda a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{p,1} & \lambda a_{p,2} & \dots & \lambda a_{p,n} \end{pmatrix}$$

Proposition 5.3

$(\mathcal{M}_{p,n}(\mathbb{K}),+,.)$ is a finite-dimensional $\mathbb{K}$-vector space with $\dim(\mathcal{M}_{p,n}(\mathbb{K})) = pn$. Moreover the family of matrices $\mathcal{B} = (E_{1,1}, E_{1,2}, \ldots, E_{1,n}, E_{2,1}, E_{2,2}, \ldots, E_{2,n}, \ldots, E_{p,1}, E_{p,2}, \ldots, E_{p,n})$ where

$$\forall k \in \{1,2,\ldots,p\},\ \forall \ell \in \{1,2,\ldots,n\},\ E_{k,\ell} = \left(\delta_{i,j}^{k,\ell} = \left\{\begin{array}{l} 1 \text{ if } (i,j) = (k,\ell) \\ 0 \text{ otherwise} \end{array}\right.\right)_{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant n}}$$

*is a basis of $\mathcal{M}_{p,n}(\mathbb{K})$ called the **canonical basis**.*

Proof: Everything does as for $\mathbb{K}^{pn}$. Actually $\mathcal{M}_{p,n}(\mathbb{K})$ is isomorphic to $\mathbb{K}^{pn}$ (as $\mathbb{K}$-vector spaces). ■

Definition 5.4 (*Transpose*)

*Let A be a matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries $(a_{i,j})_{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant n}}$. The **transpose** of A is the matrix ${}^t\!A$ in $\mathcal{M}_{n,p}(\mathbb{K})$ with entries $(a_{j,i})_{\substack{1\leqslant i\leqslant n \\ 1\leqslant j\leqslant p}}$.*

Example: The transpose of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$ is ${}^tA = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R})$.

Remark: The transpose map ${}^t : \mathcal{M}_{p,n}(\mathbb{K}) \to \mathcal{M}_{n,p}(\mathbb{K}), A \mapsto {}^tA$ is a vector space isomorphism.

5.1.2 Multiplication of matrices

Definition 5.5 (*Multiplication of matrices*)

Let A be a matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries $(a_{i,j})_{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant n}}$ and B be a matrix in $\mathcal{M}_{n,q}(\mathbb{K})$ (where $q \in \mathbb{N}^*$) with entries $(b_{i,j})_{\substack{1\leqslant i\leqslant n \\ 1\leqslant j\leqslant q}}$. We define the product of A and B, denoted $A \times B$ or shortly AB, to be the matrix in $\mathcal{M}_{p,q}(\mathbb{K})$ with entries $(\sum_{k=1}^n a_{i,k}b_{k,j})_{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant q}}$.

$$\begin{aligned} A \times B = AB &= \underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & \ldots & a_{p,n} \end{pmatrix}}_{p\times n} \times \underbrace{\begin{pmatrix} b_{1,1} & b_{1,2} & \ldots & b_{1,q} \\ b_{2,1} & b_{2,2} & \ldots & b_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \ldots & b_{n,q} \end{pmatrix}}_{n\times q} \\ &= \underbrace{\begin{pmatrix} \sum_{k=1}^n a_{1,k}b_{k,1} & \sum_{k=1}^n a_{1,k}b_{k,2} & \ldots & \sum_{k=1}^n a_{1,k}b_{k,q} \\ \sum_{k=1}^n a_{2,k}b_{k,1} & \sum_{k=1}^n a_{2,k}b_{k,2} & \ldots & \sum_{k=1}^n a_{2,k}b_{k,q} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{p,k}b_{k,1} & \sum_{k=1}^n a_{p,k}b_{k,2} & \ldots & \sum_{k=1}^n a_{p,k}b_{k,q} \end{pmatrix}}_{p\times q} \end{aligned}$$

Remark: For a row vector $R \in \mathcal{M}_{1,n}(\mathbb{K})$ and a column vector $C \in \mathcal{M}_{n,1}(\mathbb{K})$, we get

$$RC = \begin{pmatrix} r_1 & r_2 & \ldots & r_n \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \sum_{k=1}^n r_k c_k = r_1c_1 + r_2c_2 + \cdots + r_nc_n \in \mathbb{K}$$

Consequently if we write $A \in \mathcal{M}_{p,n}(\mathbb{K})$ as a list of its rows and $B \in \mathcal{M}_{n,q}(\mathbb{K})$ as a list of its columns, we get

$$\begin{cases} A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_p \end{pmatrix} \quad \text{where } \forall i \in \{1, 2, \dots p\},\ R_i = \begin{pmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \end{pmatrix} \in \mathcal{M}_{1,n}(\mathbb{K}) \\ B = \begin{pmatrix} C_1 & C_2 & \dots & C_q \end{pmatrix} \quad \text{where } \forall j \in \{1, 2, \dots q\},\ C_j = \begin{pmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{n,j} \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{K}) \end{cases}$$

and thus

$$AB = \begin{pmatrix} R_1C_1 & R_1C_2 & \dots & R_1C_q \\ R_2C_1 & R_2C_2 & \dots & R_2C_q \\ \vdots & \vdots & \ddots & \vdots \\ R_pC_1 & R_pC_2 & \dots & R_pC_q \end{pmatrix} \in \mathcal{M}_{p,q}(\mathbb{K})$$

Example : For instance, consider the following matrices:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}) \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 6 & 2 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$$

Then we get

$$\begin{aligned} AB &= \begin{pmatrix} 1\times1+3\times0+5\times1 & 1\times0+3\times6+5\times3 & 1\times2+3\times2+5\times4 & 1\times1+3\times3+5\times1 \\ 2\times1+1\times0+0\times1 & 2\times0+1\times6+0\times3 & 2\times2+1\times2+0\times4 & 2\times1+1\times3+0\times1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 33 & 28 & 15 \\ 2 & 6 & 6 & 5 \end{pmatrix} \in \mathcal{M}_{2,4}(\mathbb{R}) \end{aligned}$$

Proposition 5.6

Let A be a matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ and B be a matrix in $\mathcal{M}_{n,q}(\mathbb{K})$ (where $q \in \mathbb{N}^*$). Then

$$^t(\underbrace{A \times B}_{p\times q}) = \underbrace{^tB}_{q\times n} \times \underbrace{^tA}_{n\times p} \in \mathcal{M}_{q,p}(\mathbb{K})$$

Proof : Denote respectively $(a_{i,j})_{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant n}}$ and $(b_{i,j})_{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant q}}$ the entries of A and B. For every $i \in \{1, 2, \dots, q\}$ and $j \in \{1, 2, \dots, p\}$ the $(i,j)^{\text{th}}$ entry of the matrix $^t(AB)$ is

$$\sum_{k=1}^{n} a_{j,k} b_{k,i}$$

But the $(i,j)^{\text{th}}$ entry of the matrix $^tB^tA$ is also

$$\sum_{k=1}^{n} b_{k,i} a_{j,k} = \sum_{k=1}^{n} a_{j,k} b_{k,i}$$

The conclusion follows. ∎

5.1.3 The ring $\mathcal{M}_n(\mathbb{K})$

Definition 5.7 (*Square matrix*)

*A **square matrix** is a matrix with the same number of rows and columns. We denote $\boldsymbol{\mathcal{M}_n}(\mathbb{K})$ the set of all square matrices with dimension $n \times n$ and entries in $\mathbb{K}$.*

Remark: In case $n = p = q$ in Definition 5.5, $\times$ provides a binary operation on $\mathcal{M}_n(\mathbb{K})$.

Proposition 5.8

$(\mathcal{M}_n(\mathbb{K}), +, \times)$ is an unital ring whose identity elements are respectively the zero matrix 0_n for addition and I_n for multiplication, where

$$0_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{K}) \quad \textit{and} \quad I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$$

Proof:

1. $(\mathcal{M}_n(\mathbb{K}), +)$ is an abelian group.
2. Let A, B and C be three matrices in $\mathcal{M}_n(\mathbb{K})$ with entries respectively $(a_{i,j})_{1 \leqslant i,j \leqslant n}$, $(b_{i,j})_{1 \leqslant i,j \leqslant n}$ and $(c_{i,j})_{1 \leqslant i,j \leqslant n}$. For every indices i and j in $\{1, 2, \dots, n\}$ the $(i,j)^{\text{th}}$ entry of the matrix $(AB)C$ is
$$\sum_{\ell=1}^{n} \left(\sum_{k=1}^{n} a_{i,k} b_{k,\ell} \right) c_{\ell,j} = \sum_{1 \leqslant k,\ell \leqslant n} a_{i,k} b_{k,\ell} c_{\ell,j}$$
And the $(i,j)^{\text{th}}$ entry of the matrix $A(BC)$ is
$$\sum_{k=1}^{n} a_{i,k} \left(\sum_{\ell=1}^{n} b_{k,\ell} c_{\ell,j} \right) = \sum_{1 \leqslant k,\ell \leqslant n} a_{i,k} b_{k,\ell} c_{\ell,j}$$
Consequently $(AB)C = A(BC)$ and $\times$ is associative.
3. Let A, B and C be three matrices in $\mathcal{M}_n(\mathbb{K})$ with entries respectively $(a_{i,j})_{1 \leqslant i,j \leqslant n}$, $(b_{i,j})_{1 \leqslant i,j \leqslant n}$ and $(c_{i,j})_{1 \leqslant i,j \leqslant n}$. For every indices i and j in $\{1, 2, \dots, n\}$ the $(i,j)^{\text{th}}$ entry of the matrix $A(B + C)$ is
$$\sum_{k=1}^{n} a_{i,k}(b_{k,j} + c_{k,j}) = \sum_{k=1}^{n} a_{i,k} b_{k,j} + \sum_{k=1}^{n} a_{i,k} c_{k,j}$$
Consequently $A(B + C) = AB + AC$ and the same goes for $(B + C)A = BA + CA$.Thus, $\times$ is distributive over $+$ on $\mathcal{M}_n(\mathbb{K})$.
4. Let A be a matrix in $\mathcal{M}_n(\mathbb{K})$ with entries $(a_{i,j})_{1 \leqslant i,j \leqslant n}$. For every indices i and j in $\{1, 2, \dots, n\}$ the $(i,j)^{\text{th}}$ entry of the matrix AI_n is
$$\begin{aligned} & \sum_{k=1}^{n} a_{i,k} \delta_{k,j} \quad \text{where } \delta_{k,j} = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ otherwise} \end{cases} \\ = \; & a_{i,1} \times 0 + \dots + a_{i,j-1} \times 0 + a_{i,j} \times 1 + a_{i,j+1} \times 0 + \dots + a_{i,n} \times 0 \\ = \; & a_{i,j} \end{aligned}$$
Consequently $AI_n = A$ and the same goes for $I_n A = A$. Thus, the matrix I_n is the multiplicative identity for $\times$ in $\mathcal{M}_n(\mathbb{K})$.

Finally, $\mathcal{M}_n(\mathbb{K})$ satisfies all conditions to be an unital ring. ∎

Remarks :

- But $\times$ is not commutative on $\mathcal{M}_n(\mathbb{K})$ (as soon as $n \geqslant 2$). For instance, we have:

$$E_{1,n} \times E_{n,1} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = E_{1,1}$$

but

$$E_{n,1} \times E_{1,n} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 0_n$$

- Moreover $(\mathcal{M}_n(\mathbb{K}), +, \times)$ is not a field (as soon as $n \geqslant 2$) since there exist some zero divisors (for instance $E_{n,1}$ and $E_{1,n}$) which can not have multiplicative inverse.
- Actually we have:

$$\forall (i,j,k,\ell) \in \{1,2,\dots,n\}^4,\ E_{i,j} \times E_{k,\ell} = \delta_{j,k}.E_{i,\ell} \quad \text{where } \delta_{j,k} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ otherwise} \end{cases}$$

Definition 5.9 (*General linear group*)

We denote $\boldsymbol{\mathcal{G}\ell_n}(\mathbb{K})$ *the set of all square matrices in* $\mathcal{M}_n(\mathbb{K})$ *which have a multiplicative inverse.*

$$\mathcal{G}\ell_n(\mathbb{K}) = \{A \in \mathcal{M}_n(\mathbb{K}) \ /\ \exists B \in \mathcal{M}_n(\mathbb{K}),\ AB = BA = I_n\}$$

Example : $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \in \mathcal{G}\ell_2(\mathbb{R})$ with $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

Remark : $(\mathcal{G}\ell_n(\mathbb{K}), \times)$ is a group but a sum of matrices in $\mathcal{G}\ell_n(\mathbb{K})$ is not always in $\mathcal{G}\ell_n(\mathbb{K})$.

5.2 Matrices associated to vectors and linear maps

In this section, fix two finite-dimensional $\mathbb{K}$-vector spaces V and W.

5.2.1 Coordinate vector

Example for the coordinate space $V = \mathbb{K}^n$:

Let $v = (x_1, x_2, \dots, x_n)$ be a vector in $\mathbb{K}^n$. Since $\mathbb{K}^n$ is isomorphic to $\mathcal{M}_{n,1}(\mathbb{K})$ (both vector spaces are of dimension n), we may associate to v the following column vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{K})$$

Equivalently if $\mathcal{B} = (e_1, e_2, \dots, e_n)$ is the canonical basis of $\mathbb{K}^n$ then

$$\begin{cases} v = x_1.e_1 + x_2.e_2 + \dots + x_n.e_n \\ X = x_1.E_{1,1} + x_2.E_{2,1} + \dots + x_n.X_{n,1} \end{cases}$$

Example : For instance we may associate to the vector $v = (2, 4, 3) \in \mathbb{R}^3$ the following column vector

$$X = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \in \mathcal{M}_{3,1}(\mathbb{R})$$

with respect to the canonical basis $\mathcal{B} = \Big(e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\Big)$ of $\mathbb{R}^3$.

But notice that $\mathcal{B}' = \Big(v_1 = (0, 0, 1), v_2 = (1, 0, -1), v_3 = (1, 2, 3)\Big)$ is also a basis of $\mathbb{R}^3$ and we have $v = -3.v_1 + 0.v_2 + 2.v_3$. So we may also associate to the vector v the following column vector

$$Y = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \in \mathcal{M}_{3,1}(\mathbb{R})$$

with respect to the basis $\mathcal{B}'$.

Definition 5.10 (*Coordinate vector*)

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be a basis of V and v be a vector in V. Then v may be written uniquely as a linear combination of vectors in $\mathcal{B}$:

$$\exists!(x_1, x_2, \ldots, x_n) \in \mathbb{K}^n \ / \ v = x_1.v_1 + x_2.v_2 + \cdots + x_n.v_n$$

*The **coordinate vector of v in $\mathcal{B}$**, denoted $\mathbf{Mat}_{\mathcal{B}}(v)$, is the following column vector*

$$\mathrm{Mat}_{\mathcal{B}}(v) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{K}) \quad \textit{where } n = \dim(V)$$

Remark : For every basis $\mathcal{B}$ of V, the map $\mathrm{Mat}_{\mathcal{B}} : V \to \mathcal{M}_{n,1}(\mathbb{K})$ is a vector space isomorphism.

5.2.2 Matrix associated to a linear map

Recall that a linear map $f : V \to W$ is uniquely determined by the image of a basis of V.

Definition 5.11 (*Matrix associated to a linear map*)

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ and $\mathcal{C} = (w_1, w_2, \ldots, w_p)$ be two bases of respectively V and W and $f : V \to W$ be a linear map. Then for every $j \in \{1, 2, \ldots, n\}$, $f(v_j)$ may be written as a linear combination of vectors in $\mathcal{C}$:

$$\forall j \in \{1, 2, \ldots, n\}, \ \exists!(\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{p,j}) \in \mathbb{K}^p \ / \ f(v_j) = \lambda_{1,j}.w_1 + \lambda_{2,j}.w_2 + \cdots + \lambda_{p,j}.w_p$$

*The **matrix associated to f in $\mathcal{B}$ and $\mathcal{C}$**, denoted $\mathbf{Mat}_{\mathcal{C},\mathcal{B}}(f)$, is the following matrix*

$$\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) = (\lambda_{i,j})_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant n}} = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \ldots & \lambda_{1,n} \\ \lambda_{2,1} & \lambda_{2,2} & \ldots & \lambda_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p,1} & \lambda_{p,2} & \ldots & \lambda_{p,n} \end{pmatrix} \in \mathcal{M}_{p,n}(\mathbb{K}) \quad \textit{where} \ \begin{cases} n = \dim(V) \\ p = \dim(W) \end{cases}$$

Remark : If we write $\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f)$ as a list of its columns, we get

$$\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) = \begin{pmatrix} \mathrm{Mat}_{\mathcal{C}}(f(v_1)) & \mathrm{Mat}_{\mathcal{C}}(f(v_2)) & \dots & \mathrm{Mat}_{\mathcal{C}}(f(v_n)) \end{pmatrix} \in \mathcal{M}_{p,n}(\mathbb{K})$$

where

$$\forall j \in \{1, 2, \dots, n\},\ \mathrm{Mat}_{\mathcal{C}}(f(v_j)) \in \mathcal{M}_{p,1}(\mathbb{K})$$

Examples :

a) Consider the following linear map from the real space $V = \mathbb{R}^3$ provided with its canonical basis $\mathcal{B}$ to the real plane $W = \mathbb{R}^2$ provided with its canonical basis $\mathcal{C}$

$$f : v = (x, y, z) \longmapsto f(v) = (2x - y, 3z)$$

Then

$$\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$$

b) Let $r_{\pi/2}$ be the rotation by the angle $\frac{\pi}{2}$ counterclockwise about the origin in the real plane $V = \mathbb{R}^2$ provided with its canonical basis $\mathcal{B} = (e_1, e_2)$. Then $r_{\pi/2}(e_1) = e_2$ and $r_{\pi/2}(e_2) = -e_1$, that is

$$\begin{array}{rccc} r_{\pi/2} : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & v = (x, y) & \longmapsto & f(v) = (-y, x) \end{array}$$

And

$$\mathrm{Mat}_{\mathcal{B},\mathcal{B}}(r_{\pi/2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$$

More generally, if r_θ is the rotation by the angle $\theta \in \mathbb{R}$ counterclockwise about the origin then

$$R_\theta = \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(r_\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$$

c) Let ∂ be the differentiation map $P \mapsto P'$ from $V = \mathbb{R}_4[X]$ to itself. In the basis $\mathcal{B} = (1, X, X^2, X^3, X^4)$ we have:

$$\mathrm{Mat}_{\mathcal{B},\mathcal{B}}(\partial) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{5,5}(\mathbb{R})$$

Remark : The matrix associated to the identity function Id_V from V to itself in any basis $\mathcal{B}$ of V is

$$\mathrm{Mat}_{\mathcal{B},\mathcal{B}}(\mathrm{Id}_V) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I_n \in \mathcal{M}_n(\mathbb{K}) \quad \text{where } n = \dim(V)$$

Proposition 5.12

Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of respectively V and W. Let $f : V \to W$ be a linear map and v be a vector in V. Then

$$\underbrace{\mathrm{Mat}_{\mathcal{C}}(f(v))}_{\dim(W)\times 1} = \underbrace{\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f)}_{\dim(W)\times\dim(V)} \times \underbrace{\mathrm{Mat}_{\mathcal{B}}(v)}_{\dim(V)\times 1}$$

Proof : Denote by $(v_1, v_2, \ldots, v_n)$ the vectors in $\mathcal{B}$, by $(w_1, w_2, \ldots, w_p)$ the vectors in $\mathcal{C}$, by $(\lambda_{i,j})_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant n}}$ the entries of $\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f)$ and by $(x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$ the coordinates of the vector v written in the basis $\mathcal{B}$. Then

$$\begin{aligned}
f(v) &= f(x_1.v_1 + x_2.v_2 + \cdots + x_n.v_n)\\
&= x_1.f(v_1) + x_2.f(v_2) + \cdots + x_n.f(v_n)\\
&= \sum_{k=1}^{n} x_k.f(v_k)\\
&= \sum_{k=1}^{n} x_k.(\lambda_{1,k}.w_1 + \lambda_{2,k}.w_2 + \cdots + \lambda_{p,k}.w_p)\\
&= \sum_{k=1}^{n} \Big((x_k\lambda_{1,k}).w_1 + (x_k\lambda_{2,k}).w_2 + \cdots + (x_k\lambda_{p,k}).w_p\Big)\\
&= \left(\sum_{k=1}^{n} \lambda_{1,k}x_k\right).w_1 + \left(\sum_{k=1}^{n} \lambda_{2,k}x_k\right).w_2 + \cdots + \left(\sum_{k=1}^{n} \lambda_{p,k}x_k\right).w_p
\end{aligned}$$

The result follows. ■

Examples :

a) The image of $v = (x, y) \in \mathbb{R}^2$ under rotation by the angle $\theta \in \mathbb{R}$ counterclockwise about the origin is $r_\theta(v) = (\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y)$ since

$$\begin{aligned}
\mathrm{Mat}_{\mathcal{B}}(r_\theta(v)) &= \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(r_\theta)\,\mathrm{Mat}_{\mathcal{B}}(v)\\
&= R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta)x - \sin(\theta)y \\ \sin(\theta)x + \cos(\theta)y \end{pmatrix}
\end{aligned}$$

b) The derivative polynomial of $P(X) = -7 + 8X - 5X^2 + 2X^4 \in \mathbb{R}_4[X]$ may be computed as follows

$$\mathrm{Mat}_{\mathcal{B}}(P') = \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(\partial)\,\mathrm{Mat}_{\mathcal{B}}(P) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -7 \\ 8 \\ -5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -10 \\ 0 \\ 8 \\ 0 \end{pmatrix}$$

Consequently, $P'(X) = 8 - 10X + 8X^3$.

Proposition 5.13

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three bases of respectively U, V and W (where U is a finite-dimensional $\mathbb{K}$-vector space). Let $f : V \to W$ and $g : U \to V$ be two linear maps. Then $f \circ g : U \to W$ is a linear map and

$$\underbrace{\mathrm{Mat}_{\mathcal{C},\mathcal{A}}(f \circ g)}_{\dim(W)\times\dim(U)} = \underbrace{\mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f)}_{\dim(W)\times\dim(V)} \times \underbrace{\mathrm{Mat}_{\mathcal{B},\mathcal{A}}(g)}_{\dim(V)\times\dim(U)}$$

Proof: Denote by $(u_1, u_2, \ldots, u_n)$ the vectors in $\mathcal{A}$. Writing $\mathrm{Mat}_{\mathcal{B},\mathcal{A}}(g)$ as a list of its columns, we get

$$\begin{aligned}
& \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times \mathrm{Mat}_{\mathcal{B},\mathcal{A}}(g) \\
= \; & \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times \begin{pmatrix} \mathrm{Mat}_{\mathcal{B}}(g(u_1)) & \mathrm{Mat}_{\mathcal{B}}(g(u_2)) & \ldots & \mathrm{Mat}_{\mathcal{B}}(g(u_n)) \end{pmatrix} \\
= \; & \begin{pmatrix} \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times \mathrm{Mat}_{\mathcal{B}}(g(u_1)) & \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times \mathrm{Mat}_{\mathcal{B}}(g(u_2)) & \ldots & \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times \mathrm{Mat}_{\mathcal{B}}(g(u_n)) \end{pmatrix} \\
= \; & \begin{pmatrix} \mathrm{Mat}_{\mathcal{C}}(f(g(u_1))) & \mathrm{Mat}_{\mathcal{C}}(f(g(u_2))) & \ldots & \mathrm{Mat}_{\mathcal{C}}(f(g(u_n))) \end{pmatrix} \\
= \; & \begin{pmatrix} \mathrm{Mat}_{\mathcal{C}}((f \circ g)(u_1)) & \mathrm{Mat}_{\mathcal{C}}((f \circ g)(u_2)) & \ldots & \mathrm{Mat}_{\mathcal{C}}((f \circ g)(u_n)) \end{pmatrix} \\
= \; & \mathrm{Mat}_{\mathcal{C},\mathcal{A}}(f \circ g)
\end{aligned}$$

where the second equality comes from Definition 5.5 and the third equality from Proposition 5.12. ■

Examples:

a) For every angles $\theta \in \mathbb{R}$ and $\varphi \in \mathbb{R}$ we have

$$\begin{aligned}
R_\theta R_\varphi &= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) & -\cos(\theta)\sin(\varphi) - \sin(\theta)\cos(\varphi) \\ \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi) & -\sin(\theta)\sin(\varphi) + \cos(\theta)\sin(\varphi) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \\
&= R_{\theta+\varphi}
\end{aligned}$$

Consequently $r_\theta \circ r_\varphi = r_{\theta+\varphi}$ as expected.

b) We have after some computation:

$$\begin{pmatrix} 0&1&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&3&0 \\ 0&0&0&0&4 \\ 0&0&0&0&0 \end{pmatrix}^5 = \underbrace{\begin{pmatrix} 0&1&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&3&0 \\ 0&0&0&0&4 \\ 0&0&0&0&0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0&1&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&3&0 \\ 0&0&0&0&4 \\ 0&0&0&0&0 \end{pmatrix}}_{5 \text{ times}} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$$

Consequently for every polynomial $P(X)$ in $\mathbb{R}_4[X]$:

$$\mathrm{Mat}_{\mathcal{B}}(P^{(5)}) = \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(\partial^5) \times \mathrm{Mat}_{\mathcal{B}}(P) = (\mathrm{Mat}_{\mathcal{B},\mathcal{B}}(\partial))^5 \times \mathrm{Mat}_{\mathcal{B}}(P) = 0_5 \times \mathrm{Mat}_{\mathcal{B}}(P) = \mathrm{Mat}_{\mathcal{B}}(0)$$

Equivalently the 5^{th} order polynomial derivative of any polynomial in $\mathbb{R}_4[X]$ is the zero constant polynomial.

Corollary 5.14

Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of respectively V and W with dimension respectively n and p. The following properties hold

1. $\mathrm{Mat}_{\mathcal{C},\mathcal{B}} : L(V, W) \to \mathcal{M}_{p,n}(\mathbb{K})$ *is a vector space isomorphism.*
2. $\mathrm{Mat}_{\mathcal{B},\mathcal{B}} : L(V) \to \mathcal{M}_n(\mathbb{K})$ *is a vector space isomorphism and a ring isormorphism.*
3. $\mathrm{Mat}_{\mathcal{B},\mathcal{B}} : GL(V) \to \mathcal{G}\ell_n(\mathbb{K})$ *is a group isomorphism.*

Remark: Matrices allow arbitrary linear maps on finite-dimensional vector spaces to be represented in a convenient form, suitable for computation. Conversely, every matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ may be considered as a matrix associated to a linear map from $\mathbb{K}^n$ to $\mathbb{K}^p$ in the canonical bases.

5.2.3 Change of basis

Definition 5.15 (*Transition matrix*)

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ and $\mathcal{B}' = (v'_1, v'_2, \ldots, v'_n)$ be two bases of V. For every $j \in \{1, 2, \ldots, n\}$, v'_j may be written as a linear combination of vectors in $\mathcal{B}$:

$$\forall j \in \{1, 2, \ldots, n\},\ \exists!(\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{n,j}) \in \mathbb{K}^n \ /\ v'_j = \lambda_{1,j}.v_1 + \lambda_{2,j}.v_2 + \cdots + \lambda_{n,j}.v_n$$

The ***transition matrix from $\mathcal{B}$ to $\mathcal{B}'$****, denoted* $\boldsymbol{T_{\mathcal{B}\to\mathcal{B}'}}$*, is the following matrix*

$$T_{\mathcal{B}\to\mathcal{B}'} = (\lambda_{i,j})_{1\leqslant i,j\leqslant n} = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \ldots & \lambda_{1,n} \\ \lambda_{2,1} & \lambda_{2,2} & \ldots & \lambda_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n,1} & \lambda_{n,2} & \ldots & \lambda_{n,n} \end{pmatrix} \in \mathcal{M}_n(\mathbb{K}) \quad \textit{where } n = \dim(V)$$

Remark : If we write $T_{\mathcal{B}\to\mathcal{B}'}$ as a list of its columns, we get

$$T_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} \mathrm{Mat}_{\mathcal{B}}(v'_1) & \mathrm{Mat}_{\mathcal{B}}(v'_2) & \ldots & \mathrm{Mat}_{\mathcal{B}}(v'_n) \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$$

Example : Consider the canonical basis $\mathcal{B} = \Big(e_1 = (1,0,0), e_2 = (0,1,0),\ e_3 = (0,0,1)\Big)$ and the basis $\mathcal{B}' = \Big(v_1 = (0,0,1), v_2 = (1,0,-1), v_3 = (1,2,3)\Big)$ of the real space $V = \mathbb{R}^3$. Then

$$T_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

Moreover we have:

$$\left\{\begin{array}{lcl} v_1 & = & 0.e_1 + 0.e_2 + 1.e_3 \\ v_2 & = & 1.e_1 + 0.e_2 - 1.e_3 \\ v_3 & = & 1.e_1 + 2.e_2 + 3.e_3 \end{array}\right. \iff \left\{\begin{array}{lcl} v_1 & = & e_3 \\ v_2 & = & e_1 - e_3 \\ v_3 & = & e_1 + 2.e_2 + 3.e_3 \end{array}\right.$$

$$\iff \left\{\begin{array}{lcl} e_3 & = & v_1 \\ e_1 & = & v_2 + e_3 = v_2 + v_1 \\ e_2 & = & \frac{1}{2}.(v_3 - e_1 - 3.e_3) = \frac{1}{2}.(v_3 - v_2 - v_1 - 3.v_1) \end{array}\right.$$

$$\iff \left\{\begin{array}{lcl} e_1 & = & 1.v_1 + 1.v_2 + 0.v_3 \\ e_2 & = & -2.v_1 - \frac{1}{2}.v_2 + \frac{1}{2}.v_3 \\ e_2 & = & 1.v_1 + 0.v_2 + 0.v_3 \end{array}\right.$$

Consequently

$$T_{\mathcal{B}'\to\mathcal{B}} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

Remark that

$$T_{\mathcal{B}\to\mathcal{B}'} \times T_{\mathcal{B}'\to\mathcal{B}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$T_{\mathcal{B}\to\mathcal{B}'} \times T_{\mathcal{B}'\to\mathcal{B}} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Proposition 5.16

Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of V with dimension n. The following properties hold

1. $T_{\mathcal{B}\to\mathcal{B}'} = \mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(\mathrm{Id}_V)$
2. $T_{\mathcal{B}\to\mathcal{B}'} \in \mathcal{G}\ell_n(\mathbb{K})$ *and* $(T_{\mathcal{B}\to\mathcal{B}'})^{-1} = T_{\mathcal{B}'\to\mathcal{B}}$

Proof: 1. That follows from Definition 5.11.

2. From Proposition 5.13, we get

$$T_{\mathcal{B}\to\mathcal{B}'} \times T_{\mathcal{B}'\to\mathcal{B}} = \mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(\mathrm{Id}_V) \times \mathrm{Mat}_{\mathcal{B}',\mathcal{B}}(\mathrm{Id}_V) = \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(\mathrm{Id}_V) = I_n$$

And the same goes for $T_{\mathcal{B}'\to\mathcal{B}} \times T_{\mathcal{B}\to\mathcal{B}'} = I_n$. The conclusion follows. ∎

Corollary 5.17 (*Change of basis*)

Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of V. Let $\mathcal{C}$ and $\mathcal{C}'$ be two bases of W. Let $f : V \to W$ be a linear map and v be a vector in V. Then the following properties hold

1. $\mathrm{Mat}_{\mathcal{B}'}(v) = T_{\mathcal{B}'\to\mathcal{B}} \times \mathrm{Mat}_{\mathcal{B}}(v) = (T_{\mathcal{B}\to\mathcal{B}'})^{-1} \times \mathrm{Mat}_{\mathcal{B}}(v)$
2. $\mathrm{Mat}_{\mathcal{C}',\mathcal{B}'}(f) = T_{\mathcal{C}'\to\mathcal{C}} \times \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times T_{\mathcal{B}\to\mathcal{B}'} = (T_{\mathcal{C}\to\mathcal{C}'})^{-1} \times \mathrm{Mat}_{\mathcal{C},\mathcal{B}}(f) \times T_{\mathcal{B}\to\mathcal{B}'}$

Remark: In case $f : V \to V$ is a vector space endomorphism:

$$\mathrm{Mat}_{\mathcal{B}',\mathcal{B}'}(f) = (T_{\mathcal{B}\to\mathcal{B}'})^{-1} \times \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(f) \times T_{\mathcal{B}\to\mathcal{B}'}$$

Example: Consider the linear map f from the real space $V = \mathbb{R}^3$ to itself whose associated matrix in the canonical basis $\mathcal{B}$ is

$$\mathrm{Mat}_{\mathcal{B},\mathcal{B}}(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

Equivalently, f may be defined as follows

$$\begin{array}{rrcl} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & v=(x,y,z) & \longmapsto & f(v) = (2x, 2y, -x+2y+z) \end{array}$$

Now in the basis $\mathcal{B}' = \Big(v_1 = (0,0,1), v_2 = (1,0,-1), v_3 = (1,2,3)\Big)$, we have:

$$\begin{aligned} \mathrm{Mat}_{\mathcal{B}',\mathcal{B}'}(f) &= T_{\mathcal{B}'\to\mathcal{B}} \times \mathrm{Mat}_{\mathcal{B},\mathcal{B}}(f) \times T_{\mathcal{B}\to\mathcal{B}'} \\ &= \begin{pmatrix} 1 & -2 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 1 & -2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

It follows that f may be simply described as follows

$$f(v) = x_1.v_1 + 2x_2.v_2 + 2x_3.v_3 \quad \text{where } v = x_1.v_1 + x_2.v_2 + x_3.v_3 \in \mathbb{R}^3$$

For instance, consider the vector $v = (2,4,3) \in \mathbb{R}^3$. The coordinate vector of v in $\mathcal{B}$ is

$$X = \operatorname{Mat}_{\mathcal{B}}(v) = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \in \mathcal{M}_{3,1}(\mathbb{R})$$

And the coordinate vector of v in $\mathcal{B}'$ is

$$Y = \operatorname{Mat}_{\mathcal{B}'}(v) = T_{\mathcal{B}'\to\mathcal{B}} \times \operatorname{Mat}_{\mathcal{B}}(v) = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \in \mathcal{M}_{3,1}(\mathbb{R})$$

Consequently $v = -3.v_1 + 2.v_3$ and $f(v) = -3.v_1 + 4.v_3$.

5.2.4 Rank

Definition 5.18 (*Rank*)

Fix two positive integers $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$ and let M be a matrix in $\mathcal{M}_{p,n}(\mathbb{K})$ with entries $(m_{i,j})_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant n}}$. M may be written as a list of its columns

$$M = \begin{pmatrix} C_1 & C_2 & \dots & C_n \end{pmatrix} \in \mathcal{M}_{p,n}(\mathbb{K}) \quad \textit{where } \forall j \in \{1,2,\dots,n\},\ C_j = \begin{pmatrix} m_{1,j} \\ m_{2,j} \\ \vdots \\ m_{p,j} \end{pmatrix} \in \mathcal{M}_{p,1}(\mathbb{K})$$

*The **rank** of M, denoted **rank(M)**, is the dimension of the subspace of $\mathcal{M}_{p,1}(\mathbb{K})$ spanned by the columns of M.*

$$\operatorname{rank}(M) = \dim\Big(\operatorname{Span}(\{C_1, C_2, \dots, C_n\})\Big)$$

Remark: The rank of $M \in \mathcal{M}_{p,n}(\mathbb{K})$ is less than $\dim(\mathcal{M}_{p,1}(\mathbb{K})) = p$ by definition and than n since $(C_1, C_2, \dots, C_n)$ is a spanning family of n vectors (in the subspace spanned by the columns of M). It follows that $\operatorname{rank}(M) \in \{0, 1, \dots, \min\{n,p\}\}$.

Proposition 5.19

The rank of a matrix associated to a linear map $f : V \to W$ in any bases $\mathcal{B}$ and $\mathcal{C}$ of respectively V and W is equal to the rank of f.

$$\operatorname{rank}\Big(\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f)\Big) = \operatorname{rank}(f)$$

Proof: Denote by $(v_1, v_2, \dots, v_n)$ the vectors in $\mathcal{B}$. If we write $\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f)$ as a list of its columns, we get

$$\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f) = \begin{pmatrix} \operatorname{Mat}_{\mathcal{C}}(f(v_1)) & \operatorname{Mat}_{\mathcal{C}}(f(v_2)) & \dots & \operatorname{Mat}_{\mathcal{C}}(f(v_n)) \end{pmatrix} \in \mathcal{M}_{p,n}(\mathbb{K})$$

Then

$$\begin{aligned}
\operatorname{rank}\Big(\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f)\Big) &= \dim\Big(\operatorname{Span}(\{\operatorname{Mat}_{\mathcal{C}}(f(v_1)),\operatorname{Mat}_{\mathcal{C}}(f(v_2)),\dots,\operatorname{Mat}_{\mathcal{C}}(f(v_n))\})\Big)\\
&= \dim\Big(\operatorname{Mat}_{\mathcal{C}}(f(\underbrace{\operatorname{Span}(\{v_1,v_2,\dots,v_n\})}_{V}))\Big)\\
&= \dim\Big(\operatorname{Mat}_{\mathcal{C}}(f(V))\Big)
\end{aligned}$$

Since $\operatorname{Mat}_{\mathcal{C}} : W \to \mathcal{M}_{p,1}(\mathbb{K})$ is a vector space isomorphism, it follows that

$$\operatorname{rank}\Big(\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f)\Big) = \dim\Big(\operatorname{Mat}_{\mathcal{C}}(f(V))\Big) = \dim\Big(f(V)\Big) = \dim\Big(\operatorname{Im}(f)\Big) = \operatorname{rank}(f)$$

■

Proposition 5.20

Fix two positive integers $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$ and let M be a matrix in $\mathcal{M}_{p,n}(\mathbb{K})$. The rank of M is equal to $r \in \{0,1,\dots,\min\{n,p\}\}$ if and only if there exist $P \in \mathcal{G}\ell_p(\mathbb{K})$ and $Q \in \mathcal{G}\ell_n(\mathbb{K})$ such that

$$PMQ = \underbrace{\left(\begin{array}{cccc|ccc}
1 & 0 & \dots & 0 & 0 & \dots & 0\\
0 & 1 & \dots & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 1 & 0 & \dots & 0\\ \hline
0 & 0 & \dots & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 0 & 0 & \dots & 0
\end{array}\right)}_{r \text{ columns} \quad (n-r) \text{ columns}}
\begin{array}{l}\left.\vphantom{\begin{array}{c}1\\0\\\vdots\\0\end{array}}\right\} r \text{ rows}\\ \left.\vphantom{\begin{array}{c}0\\\vdots\\0\end{array}}\right\} (p-r) \text{ rows}\end{array}
\in \mathcal{M}_{p,n}(\mathbb{K})$$

Proof: Denote by J_r the matrix above and by $f : \mathbb{K}^n \to \mathbb{K}^p$ the linear map such that $M = \operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f)$ where $\mathcal{B}$ and $\mathcal{C}$ are the canonical bases of respectively $\mathbb{K}^n$ and $\mathbb{K}^p$.

Necessary. If $r = \operatorname{rank}(M) = \operatorname{rank}(f) = \dim(\operatorname{Im}(f))$ then $\dim(\operatorname{Ker}(f)) = \dim(\mathbb{K}^n) - \dim(\operatorname{Im}(f)) = n - r$ from the rank theorem. Denote by $(v_{r+1}, v_{r+2}, \dots, v_n)$ a basis of $\operatorname{Ker}(f)$. Since this family is linearly independent, it is a subfamily of some basis of $\mathbb{K}^n$, say $\mathcal{B}' = (v_1, v_2, \dots, v_n)$. Now, consider the family $(w_1 = f(v_1), w_2 = f(v_2), \dots, w_r = f(v_r))$. By construction, it is a basis of $\operatorname{Im}(f)$, thus a subfamily of some basis of $\mathbb{K}^p$, say $\mathcal{C}' = (w_1, w_2, \dots, w_p)$. Finally we have $\operatorname{Mat}_{\mathcal{C}',\mathcal{B}'}(f) = J_r$. Consequently if $P = T_{\mathcal{C}'\to\mathcal{C}} \in \mathcal{G}\ell_p(\mathbb{K})$ and $Q = T_{\mathcal{B}\to\mathcal{B}'} \in \mathcal{G}\ell_n(\mathbb{K})$ then

$$PMQ = T_{\mathcal{C}'\to\mathcal{C}} \times \operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f) \times T_{\mathcal{B}\to\mathcal{B}'} = \operatorname{Mat}_{\mathcal{C}',\mathcal{B}'}(f) = J_r$$

Sufficient. Denote by $g : \mathbb{K}^p \to \mathbb{K}^p$ and $h : \mathbb{K}^n \to \mathbb{K}^n$ the linear maps such that $P = \operatorname{Mat}_{\mathcal{C},\mathcal{C}}(g)$ and $Q = \operatorname{Mat}_{\mathcal{B},\mathcal{B}}(h)$. Then

$$\begin{aligned}
r = \operatorname{rank}\Big(J_r\Big) &= \operatorname{rank}\Big(PMQ\Big)\\
&= \operatorname{rank}\Big(\operatorname{Mat}_{\mathcal{C},\mathcal{C}}(g) \times \operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f) \times \operatorname{Mat}_{\mathcal{B},\mathcal{B}}(h)\Big)\\
&= \operatorname{rank}\Big(\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(g \circ f \circ h)\Big)\\
&= \operatorname{rank}\Big(g \circ f \circ h\Big)\\
&= \dim\Big(\operatorname{Im}(g \circ f \circ h)\Big)
\end{aligned}$$

Moreover g and h are vector space isomorphisms since their associated matrices P and Q have multiplicative inverse. It follows that

$$\begin{aligned} r = \operatorname{rank}\Big(J_r\Big) &= \dim\Big(\operatorname{Im}(g \circ f \circ h)\Big) \\ &= \dim\Big(g(f(h(\mathbb{K}^n)))\Big) \\ &= \dim\Big(f(\mathbb{K}^n)\Big) \\ &= \dim\Big(\operatorname{Im}(f)\Big) \\ &= \operatorname{rank}\Big(f\Big) \\ &= \operatorname{rank}\Big(\operatorname{Mat}_{\mathcal{C},\mathcal{B}}(f)\Big) = \operatorname{rank}(M) \end{aligned}$$

■

Corollary 5.21

Fix a positive integer $n \in \mathbb{N}^*$ *and let* M *be a matrix in* $\mathcal{M}_n(\mathbb{K})$*. Then* $M \in \mathcal{G}\ell_n(\mathbb{K})$ *if and only if* $\operatorname{rank}(M) = n$*.*

Remark : Furthermore, every matrix in $\mathcal{G}\ell_n(\mathbb{K})$ may be considered as a transition matrix from the canonical basis of $\mathbb{K}^n$ to the basis formed with its column vectors (by identifying $\mathbb{K}^n$ and $\mathcal{M}_{n,1}(\mathbb{K})$).

Corollary 5.22

Fix two positive integers $n \in \mathbb{N}^*$ *and* $p \in \mathbb{N}^*$ *and let* M *be a matrix in* $\mathcal{M}_{p,n}(\mathbb{K})$*. Then*

$$\operatorname{rank}({}^tM) = \operatorname{rank}(M)$$

Remark : In particular, the rank of a matrix is the maximal number of its linearly independent columns or rows.