Chapter 4

Finite-dimensional vector spaces

In this chapter, fix a commutative field $(\mathbb{K}, +, \times)$ (for instance $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

4.1 Family of vectors

In this section, fix a $\mathbb{K}$-vector space V.

4.1.1 Linearly independent family

Definition 4.1 (*Linearly independent family*)

Let $(v_1, v_2, \ldots, v_n)$ be a family of vectors in V. This family is said to be ***linearly independent*** *if it satisfies the following condition*

$$\forall (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^n,\ \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n = 0_V \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0_{\mathbb{K}}$$

Conversely this family is said ***linearly dependent*** *if there exist some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that the linear combination $\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$ is equal to the zero vector.*

Examples:

a) Any family of only one non zero vector $v \in V - \{0_V\}$ is linearly independent since

$$\forall \lambda \in \mathbb{K},\ \lambda.v = 0_V \Longleftrightarrow \lambda = 0_{\mathbb{K}}$$

b) Any family of vectors containing the zero vector is linearly dependent.

Proof: Let $v_1, v_2, \ldots, v_n$ be such a family of vectors. Without loss of generality, we may reorder the family such that $v_1 = 0_V$. Now choosing any scalar $\lambda_1 \neq 0_{\mathbb{K}}$ and $\lambda_2 = \cdots = \lambda_n = 0_{\mathbb{K}}$, we get

$$\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n = \lambda_1.0_V + 0_{\mathbb{K}}.(v_2 + \cdots + v_n) = 0_V$$

■

c) Let v and v' be two non zero vectors in $V - \{0_V\}$. Then the family (v, v') is linearly dependent if and only if v and v' are colinear (that is there exists $\mu \in \mathbb{K}$ such that $v = \mu.v'$).

Proof: If v and v' are colinear then $\lambda = 1_{\mathbb{K}} \neq 0_{\mathbb{K}}$ and $\lambda' = -\mu$ give $\lambda.v + \lambda'.v' = v - \mu.v' = 0_V$. Conversely, if there exist λ and λ' not both zero in $\mathbb{K}$ such that $\lambda.v + \lambda'.v' = 0_V$ then $\lambda \neq 0_{\mathbb{K}}$ (otherwise $\lambda' \neq 0_{\mathbb{K}}$ and $\lambda'.v' = 0_V$ are a contradiction with $v' \neq 0_V$) and $\mu = -\lambda'\lambda^{-1}$ implies $v = \mu.v'$ as needed. ■

d) Consider the following family of the real space $\mathbb{R}^3$

$$\Big(v_1 = (0,0,1), v_2 = (1,0,-1), v_3 = (1,2,3)\Big)$$

It is a linearly independent family of vectors in $\mathbb{R}^3$.

Proof: For every $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, we have:

$$\begin{aligned}
\lambda_1.v_1 + \lambda_2.v_2 + \lambda_3.v_3 = 0_{\mathbb{R}^3} &\iff \lambda_1.(0,0,1) + \lambda_2.(1,0,-1) + \lambda_3.(1,2,3) = (0,0,0) \\
&\iff (\lambda_2 + \lambda_3, 2\lambda_3, \lambda_1 - \lambda_2 + 3\lambda_3) = (0,0,0) \\
&\iff \begin{cases} 2\lambda_3 &= 0 \\ \lambda_2 + \lambda_3 &= 0 \\ \lambda_1 - \lambda_2 + 3\lambda_3 &= 0 \end{cases} \\
&\iff \begin{cases} \lambda_3 = 0 \\ \lambda_2 = -\lambda_3 = 0 \\ \lambda_1 = \lambda_2 - 3\lambda_3 = 0 \end{cases}
\end{aligned}$$

The result follows. ■

Proposition 4.2

Every subfamily of a linearly independent family of vectors in V is also linearly independent.

Proof: Let $(v_1, v_2, \dots, v_n)$ be a linearly independent family of vectors in V. Consider a subfamily of this family. Without loss of generality, we may reorder the family such that the subfamily is $(v_1, v_2, \dots, v_k)$ with $k \leqslant n$. Now let $\lambda_1, \lambda_2, \dots, \lambda_k$ be scalars in $\mathbb{K}$ such that

$$\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_k.v_k = 0_V$$

Then $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0_{\mathbb{K}}$ give

$$\lambda_1.v_1+\lambda_2.v_2+\cdots+\lambda_n.v_n = \underbrace{\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_k.v_k}_{=0_V} + \underbrace{\lambda_{k+1}.v_{k+1} + \lambda_{k+2}.v_{k+2} + \cdots + \lambda_n.v_n}_{=0_{\mathbb{K}}.(v_{k+1}+v_{k+2}+\cdots+v_n)=0_V} = 0_V$$

Since $(v_1, v_2, \dots, v_n)$ is linearly independent, it follows $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0_{\mathbb{K}}$, and it particular $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0_{\mathbb{K}}$. Consequently the subfamily $(v_1, v_2, \dots, v_k)$ is also linearly independent. ■

4.1.2 Spanning family and basis

Definition 4.3 (*Spanning family*)

*Let $(v_1, v_2, \dots, v_n)$ be a family of vectors in V. This family is said to be a **spanning family** if the linear span of $\{v_1, v_2, \dots, v_n\}$ is the whole vector space V. Equivalently, $(v_1, v_2, \dots, v_n)$ is a spanning family if it satisfies the following condition*

$$\forall v \in V,\ \exists(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^n \ / \ v = \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$$

Proposition 4.4

Every superfamily of a spanning family of vectors in V is also a spanning family.

Proof: Let $(v_1, v_2, \dots, v_n)$ be a superfamily of a spanning family of vectors in V. Without loss of generality, we may reorder the family such that $(v_1, v_2, \dots, v_k)$ with $k \leqslant n$ is a spanning family. Now let v be a vector in V. Since $(v_1, v_2, \dots, v_k)$ is a spanning family, it follows that there exist some $\lambda_1, \lambda_2, \dots, \lambda_k$ scalars in $\mathbb{K}$ such that

$$v = \lambda_1.v_1 + \lambda_2 + v_2 + \cdots + \lambda_k.v_k$$

Then $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0_{\mathbb{K}}$ give

$$v = \underbrace{\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_k.v_k}_{=v} + \underbrace{\lambda_{k+1}.v_{k+1} + \lambda_{k+2}.v_{k+2} + \cdots + \lambda_n.v_n}_{=0_{\mathbb{K}}.(v_{k+1}+v_{k+2}+\cdots+v_n)=0_V} = \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$$

Consequently the superfamily $(v_1, v_2, \dots, v_n)$ is also a spanning family. ■

Remark: If $(v_1, v_2, \dots, v_n)$ is a spanning family then every vector in V may be written as a linear combination of vectors in this family with scalars $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^n$. But these scalars are not necessarily unique.

Definition 4.5 (*Basis*)

A ***basis*** *of the vector space V is a linearly independent spanning family $\mathcal{B} = (v_1, v_2, \dots, v_n)$.*

Proposition 4.6

Let $\mathcal{B} = (v_1, v_2, \dots, v_n)$ be a family of vectors in V. $\mathcal{B}$ is a basis of V if and only if every vector in V may be written uniquely as a linear combination of $v_1, v_2, \dots, v_n$. Equivalently, $\mathcal{B}$ is a basis if it satisfies the following condition

$$\forall v \in V,\ \exists!(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^n \ /\ v = \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$$

Proof: **Sufficient.** If every vector may be written uniquely as a linear combination of vectors in $\mathcal{B}$, then in particular $\mathcal{B}$ is a spanning family. Now assume the linear combination $\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$ is equal to the zero vector for some scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{K}$. Then the unicity of the writing $0_V = 0_{\mathbb{K}}.v_1 + 0_{\mathbb{K}}.v_2 + \cdots + 0_{\mathbb{K}}.v_n$ implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0_{\mathbb{K}}$. Consequently $\mathcal{B}$ is linearly independent and thus a basis of V.

Necessary. Let v be a vector in V. Since $\mathcal{B}$ is a spanning family, there exist some scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{K}$ such that $v = \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$. Now assume there exist some others scalars $\lambda'_1, \lambda'_2, \dots, \lambda'_n$ in $\mathbb{K}$ such that $v = \lambda'_1.v_1 + \lambda'_2.v_2 + \cdots + \lambda'_n.v_n$. Then

$$\begin{aligned} 0_V &= v - v \\ &= (\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n) - (\lambda'_1.v_1 + \lambda'_2.v_2 + \cdots + \lambda'_n.v_n) \\ &= (\lambda_1 - \lambda'_1).v_1 + (\lambda_2 - \lambda'_2).v_2 + \cdots + (\lambda_n - \lambda'_n).v_n \end{aligned}$$

Since $\mathcal{B}$ is linearly independent, it follows that $\lambda_1 - \lambda'_1 = \lambda_2 - \lambda'_2 = \cdots = \lambda_n - \lambda'_n = 0_{\mathbb{K}}$ or equivalently $\lambda_1 = \lambda'_1$, $\lambda_2 = \lambda'_2$, ..., and $\lambda_n = \lambda'_n$. Consequently the writing $v = \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n$ is unique. ■

Example: Consider the following family of the real space $\mathbb{R}^3$

$$\mathcal{B} = \Big(v_1 = (0,0,1), v_2 = (1,0,-1), v_3 = (1,2,3)\Big)$$

It is a basis of $\mathbb{R}^3$.

Proof: We have already shown that $\mathcal{B}$ is a linearly independent family of vectors in $\mathbb{R}^3$. Now let $v = (x, y, z)$ be a vector in $\mathbb{R}^3$. For every $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{K}^3$, we have:

$$\begin{aligned}
v = \lambda_1.v_1 + \lambda_2.v_2 + \lambda_3.v_3 &\iff (x, y, z) = \lambda_1.(0, 0, 1) + \lambda_2.(1, 0, -1) + \lambda_3.(1, 2, 3) \\
&\iff (x, y, z) = (\lambda_2 + \lambda_3, 2\lambda_3, \lambda_1 - \lambda_2 + 3\lambda_3) \\
&\iff \left\{ \begin{array}{rcl} 2\lambda_3 &=& y \\ \lambda_2 + \lambda_3 &=& x \\ \lambda_1 - \lambda_2 + 3\lambda_3 &=& z \end{array} \right. \\
&\iff \left\{ \begin{array}{l} \lambda_3 = \frac{y}{2} \\ \lambda_2 = x - \lambda_3 = x - \frac{y}{2} \\ \lambda_1 = z + \lambda_2 - 3\lambda_3 = z + x - \frac{y}{2} - \frac{3y}{2} = x - 2y + z \end{array} \right.
\end{aligned}$$

Consequently $v = (x - 2y + z).v_1 + (x - \frac{y}{2}).v_2 + \frac{y}{2}.v_3$. It follows that $\mathcal{B}$ is a spanning family and thus a basis of $\mathbb{R}^3$. ∎

4.2 Finite dimension

4.2.1 Finite-dimensional vector space

Definition 4.7 (*Finite dimensional vector space*)

Let V be a $\mathbb{K}$-vector space. V is said to be ***finite-dimensional*** *if there exist a spanning family of vectors in V.*

Examples:

a) For every positive integer $n \in \mathbb{N}^*$, the coordinate space $\mathbb{K}^n$ is finite-dimensional. Indeed the following family of vectors in $\mathbb{K}^n$ is a basis

$$\mathcal{B} = \Big(e_1, e_2, \ldots, e_n\Big) \quad \text{where } \forall k \in \{1, 2, \ldots, n\},\ e_k = (0_{\mathbb{K}}, 0_{\mathbb{K}}, \ldots, 0_{\mathbb{K}}, \underbrace{1_{\mathbb{K}}}_{k^{\text{th}} \text{ position}}, 0_{\mathbb{K}}, \ldots, 0_{\mathbb{K}})$$

This family is called the **canonical basis** of $\mathbb{K}^n$.

Proof: Linearly independent. For every $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^n$, we have:

$$\begin{aligned}
\lambda_1.e_1 + \lambda_2.e_2 + \cdots + \lambda_n.e_n = 0_{\mathbb{K}^n} &\iff (\lambda_1, \lambda_2, \ldots, \lambda_n) = (0_{\mathbb{K}}, 0_{\mathbb{K}}, \ldots, 0_{\mathbb{K}}) \\
&\iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0_{\mathbb{K}}
\end{aligned}$$

Spanning family. Let $v = (x_1, x_2, \ldots, x_n)$ be a vector in $\mathbb{K}^n$. Then

$$v = (x_1, x_2, \ldots, x_n) = x_1.e_1 + x_2.e_2 + \cdots + x_n.e_n$$

∎

b) Counterexample. The $\mathbb{K}$-vector space $\mathbb{K}[X]$ of all polynomials with coefficients in $\mathbb{K}$ is not finite-dimensional.

Proof: Assume there exists a spanning family, say $(P_1(X), P_2(X), \ldots, P_n(X))$. Then every polynomial $P(X) \in \mathbb{K}[X]$ may be written as $P(X) = \lambda_1.P_1(X) + \lambda_2.P_2(X) + \cdots + \lambda_n.P_n(X)$ where $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^n$. Consequently

$$\begin{aligned}
\deg(P) &= \deg(\lambda_1.P_1 + \lambda_2.P_2 + \cdots + \lambda_n.P_n) \\
&\leqslant \max\{\lambda_1.P_1, \lambda_2.P_2, \ldots, \lambda_n.P_n\} \\
&\leqslant \max\{P_1, P_2, \ldots, P_n\}
\end{aligned}$$

But $P(X) = X^{d+1}$ with $d = \max\{P_1, P_2, \dots, P_n\} \geqslant 0$ gives $d + 1 \leqslant d$ which is a contradiction. ■

c) But the subspace $\mathbb{K}_d[X]$ of all polynomials with degree at most a given integer d is finite-dimensional. Indeed the family $(1, X, X^2, \dots, X^d)$ of successive powers of X is a basis (it is linearly independent from the definition of polynomials and a spanning family from the exact Taylor's formula).

Proposition 4.8

Let V be a finite-dimensional $\mathbb{K}$-vector space. Then the following properties hold

1. *Every linearly independent family of vectors in V is a subfamily of some basis of V.*
2. *Every spanning family of vectors in V contains some basis of V as subfamily.*

Proof: Let $\mathcal{F}$ be a spanning family of vectors in V and n be the number of vectors in this family.

1. Let $\mathcal{F}_0 = (v_1, v_2, \dots, v_m)$ be a linearly independent family of vectors in V. Now denote by $(v_{m+1}, v_{m+2}, \dots, v_{m+n})$ the vectors in $\mathcal{F}$ and by $\mathcal{F}_n$ the family $(v_1, v_2, \dots, v_{m+n})$ which is a spanning family from Proposition 4.4.

$$\mathcal{F}_0 = (\underbrace{v_1, v_2, \dots, v_m}_{\text{linearly independent}}) \subset \mathcal{F}_n = (\underbrace{\underbrace{v_1, v_2, \dots, v_m}_{\text{linearly independent}}, \underbrace{v_{m+1}, v_{m+2}, \dots, v_{m+n}}_{\text{spanning family}}}_{\text{spanning family}})$$

If $\mathcal{F}_0$ is a spanning family then it is a basis and the conclusion holds. Thus assume that $\mathcal{F}_0$ is not a spanning family. Consequently there exists at least one vector in $(v_{m+1}, v_{m+2}, \dots, v_{m+n})$ which may not be written as a linear combination of vectors in $\mathcal{F}_0$ (otherwise the linear span $\mathrm{Span}(v_{m+1}, v_{m+2}, \dots, v_{m+n}) = V$ would be included in $\mathrm{Span}(v_1, v_2, \dots, v_m) \subsetneq V$ which is a contradiction). Without loss of generality, we may reorder the family $(v_{m+1}, v_{m+2}, \dots, v_{m+n})$ such that v_{m+1} may not be written as a linear combination of vectors in $\mathcal{F}_0$. Now denote by $\mathcal{F}_1$ the family $(v_1, v_2, \dots, v_m, v_{m+1})$. We are going to prove that $\mathcal{F}_1$ is linearly independent. Assume $\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_{m+1}.v_{m+1} = 0_V$ for some scalars $(\lambda_1, \lambda_2, \dots, \lambda_{m+1}) \in \mathbb{K}^{m+1}$.

First case. $\lambda_{m+1} \neq 0_{\mathbb{K}}$
Then we get $v_{m+1} = (-\lambda_{m+1}^{-1}\lambda_1).v_1 + (-\lambda_{m+1}^{-1}\lambda_2).v_2 + \cdots + (-\lambda_{m+1}^{-1}\lambda_m).v_m$ that is v_{m+1} is a linear combination of vectors in $\mathcal{F}_0$ which is a contradiction.

Second case. $\lambda_{m+1} = 0_{\mathbb{K}}$
Then $\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_m.v_m = 0_V$ implies $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0_{\mathbb{K}}$ since $\mathcal{F}_0$ is linearly independent.

Finally $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda_{m+1} = 0_{\mathbb{K}}$ and thus $\mathcal{F}_1$ is linearly independent.

$$\mathcal{F}_1 = (\underbrace{v_1, v_2, \dots, v_{m+1}}_{\text{linearly independent}}) \subset \mathcal{F}_n = (\underbrace{v_1, v_2, \dots, v_m, v_{m+1}, v_{m+2}, \dots, v_{m+n}}_{\text{spanning family}})$$

If $\mathcal{F}_1$ is a spanning family then it is a basis which contains the starting linearly independent family $\mathcal{F}_0$ and the conclusion holds. Otherwise we may iterate the previous process to construct an increasing sequence of linearly independent families

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$$

This algorithm stops as soon as one constructed family $\mathcal{F}_k$ for $k \in \{0, 1, 2, \dots, n\}$ is a spanning family (at worst for $k = n$). Then $\mathcal{F}_k$ is a basis which contains the starting linearly independent family $\mathcal{F}_0$ and the conclusion holds.

2. Let $\mathcal{F}_n = \mathcal{F} = (v_1, v_2, \ldots, v_n)$ be the starting spanning family. Without loss of generality, we may reorder this family such that $v_1 \neq 0_V$. Now denote by $\mathcal{F}_1$ the family (v_1) of only one vector which is linearly independent.

$$\mathcal{F}_1 = (\underbrace{v_1}_{\text{linearly independent}}) \subset \mathcal{F}_n = (\underbrace{v_1, v_2, \ldots, v_n}_{\text{spanning family}})$$

Then we may use the same algorithm as the previous point to construct an increasing sequence of linearly independent families

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$$

When the algorithm stops, we get a basis $\mathcal{F}_k$ with $k \in \{0, 1, 2, \ldots, n\}$ which is contained in the starting spanning family $\mathcal{F}_n$ and the conclusion holds. ■

Remark : In particular, every finite-dimensional $\mathbb{K}$-vector space has a basis. Indeed, any family of only one non zero vector is linearly independent thus is contained in some basis.

4.2.2 Dimension

Theorem 4.9 (*Dimension theorem*)

Let V be a finite-dimensional $\mathbb{K}$-vector space. Then all bases of V have the same cardinality.

Proof : At first, we are going to prove the following result:

Lemma 4.10

If $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ is a basis of V and $\mathcal{F} = (w_1, w_2, \ldots, w_m)$ is a spanning family of vectors in V then $n \leqslant m$.

Proof of the lemma : Since $\mathcal{F}$ is a spanning family, the vector v_1 may be written as a linear combination of vectors in $\mathcal{F}$.

$$\exists(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{K}^m \ / \ v_1 = \lambda_1.w_1 + \lambda_2.w_2 + \cdots + \lambda_m.w_m$$

Since $\mathcal{B}$ is linearly independent, v_1 is not equal to the zero vector and the scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ are not all zero. Without loss of generality, we may reorder $\mathcal{F}$ such that $\lambda_1 \neq 0_{\mathbb{K}}$. Then

$$w_1 = \lambda_1^{-1}.v_1 + (-\lambda_1^{-1}\lambda_2).w_2 + \cdots + (-\lambda_1^{-1}\lambda_m).w_m$$

It follows that the linear span $\mathrm{Span}(w_1, w_2, \ldots, w_m) = V$ is included in $\mathrm{Span}(v_1, w_2, \ldots, w_m)$ and then $\mathcal{F}_1 = (v_1, w_2, \ldots, w_m)$ is a spanning family.

By induction, we may show as well that $\mathcal{F}_k = (v_1, \ldots, v_k, w_{k+1}, w_{k+2}, \ldots, w_m)$ is a spanning family for any $k \in \{1, 2, \ldots, n\}$ (after some possible reordering of $\mathcal{F}$ if necessary). Indeed assume $\mathcal{F}_k$ is a spanning family for some $k \in \{1, 2, \ldots, n-1\}$. Then the vector v_{k+1} may be written as a linear combination of vectors in $\mathcal{F}_k$.

$$\begin{aligned} \exists(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{K}^m \ / \quad & v_{k+1} = \lambda_1.v_1 + \cdots + \lambda_k.v_k + \lambda_{k+1}.w_{k+1} + \cdots + \lambda_m.w_m \\ & \iff (-\lambda_1).v_1 + \cdots + (-\lambda_k).v_k + v_{k+1} = \lambda_{k+1}.w_{k+1} + \cdots + \lambda_m.w_m \end{aligned}$$

Since $\mathcal{B}$ is linearly independent, $(-\lambda_1).v_1 + \cdots + (-\lambda_k).v_k + v_{k+1}$ is not equal to the zero vector and the scalars $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m$ are not all zero. Without loss of generality, we may reorder $(w_{k+1}, w_{k+2}, \ldots, w_m)$ such that $\lambda_{k+1} \neq 0_{\mathbb{K}}$. Then

$$w_{k+1} = (-\lambda_{k+1}^{-1}\lambda_1).v_1 + \cdots + (-\lambda_{k+1}^{-1}\lambda_k).v_k + \lambda_{k+1}^{-1}.v_{k+1} + (-\lambda_{k+1}^{-1}\lambda_{k+2}).w_{k+2} + \cdots + (-\lambda_{k+1}^{-1}\lambda_m).w_m$$

It follows that the linear span $\mathrm{Span}(v_1, \ldots, v_k, w_{k+1}, w_{k+2}, \ldots, w_m) = V$ (by the inductive hypothesis) is included in $\mathrm{Span}(v_1, \ldots, v_k, v_{k+1}, w_{k+2}, \ldots, w_m)$ and then $\mathcal{F}_{k+1}$ is a spanning family. That concludes the induction.

In particular, we get that $\mathcal{F}_n = (v_1, \ldots, v_n, w_{n+1}, w_{n+2}, \ldots, w_m)$ is a spanning family and that $n \leqslant m$. ■

Now let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of V. From Lemma 4.10, we get $\mathrm{Card}(\mathcal{B}) \leqslant \mathrm{Card}(\mathcal{B}')$ since $\mathcal{B}'$ is a spanning family and $\mathrm{Card}(\mathcal{B}') \leqslant \mathrm{Card}(\mathcal{B})$ since $\mathcal{B}$ is a spanning family. Consequently $\mathcal{B}$ and $\mathcal{B}'$ have the same cardinality and the conclusion follows. ■

Definition 4.11 (*Dimension*)

Let V be a finite-dimensional $\mathbb{K}$-vector space. The **dimension** *of V, denoted* $\mathbf{dim(}\boldsymbol{V}\mathbf{)}$, *is the number of vectors in any basis of V.*

Examples:

a) For every $n \in \mathbb{N}^*$, $\dim(\mathbb{K}^n) = n$ since the canonical basis of $\mathbb{K}^n$ contains exactly n vectors.

b) $\dim(\{0_V\}) = 0$ since $\emptyset$ is a basis of the trivial vector space.

c) For any given integer $d \in \mathbb{N}$, $\dim(\mathbb{K}_d[X]) = d+1$ since the basis $(1, X, X^2, \ldots, X^d)$ contains exactly $d+1$ polynomials.

Proposition 4.12

Let V be a finite-dimensional $\mathbb{K}$-vector space. The the following properties hold

1. *Every linearly independent family contains at most* $\dim(V)$ *vectors and is a basis if and only if it contains exactly* $\dim(V)$ *vectors.*
2. *Every spanning family contains at least* $\dim(V)$ *vectors and is a basis if and only if it contains exactly* $\dim(V)$ *vectors.*

Proof: That follows from Proposition 4.8 and Dimension theorem 4.9. ■

Example: For any given integer $d \in \mathbb{N}$ and any element $a \in \mathbb{K}$, the family

$$\mathcal{B} = \Big(1, (X-a), (X-a)^2, \ldots, (X-a)^d\Big)$$

is a basis of $\mathbb{K}_d[X]$. Indeed it is a spanning family from the exact Taylor's formula and it contains exactly $d+1 = \dim(\mathbb{K}_d[X])$ polynomials.

4.2.3 Finite-dimensional subspace

Proposition 4.13

Let V be a finite-dimensional $\mathbb{K}$-vector space and W be a subspace of V. Then W is also finite-dimensional and

$$\dim(W) \leqslant \dim(V)$$

Furthermore

$$\dim(W) = \dim(V) \Longleftrightarrow W = V$$

Proof: At first, we prove that W is finite-dimensional. Let $\mathcal{F}_1 = (w_1)$ be a family of only one non zero vector in W. In particular, $\mathcal{F}_1$ is linearly independent. If $\mathcal{F}_1$ is a spanning family of W then it is a basis of W and the conclusion holds. Thus assume that $\mathcal{F}_1$ is not a spanning family of W. Consequently there exists at least one vector in W, say w_2, which may not be written as a linear combination of vectors in $\mathcal{F}_1$ (that is w_2 is not colinear to w_1). As in the proof of Proposition 4.8, we deduce that the family $\mathcal{F}_2 = (w_1, w_2)$ is linearly independent. If $\mathcal{F}_2$ is a spanning family of W then it is a basis and the conclusion holds. Otherwise we may iterate the previous process to construct an increasing sequence of linearly independent families

$$\mathcal{F}_1 = (w_1) \subset \mathcal{F}_2 = (w_1, w_2) \subset \cdots \subset \mathcal{F}_k = (w_1, w_2, \ldots, w_k) \subset \ldots$$

This algorithm stops as soon as one constructed family $\mathcal{F}_k$ for $k \geqslant 1$ is a spanning family. It happens at worst for $k = \dim(V)$, since we would get a basis of V (from Proposition 4.12) and thus a spanning family of $W \subset V$. Consequently, W is finite-dimensional.

The remaining follows from Proposition 4.12. ■

Proposition 4.14

Let W and W' be two subspaces of a finite-dimensional $\mathbb{K}$-vector space V. Then

$$\dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W')$$

In particular

$$\left\{ \begin{array}{l} \dim(V) = \dim(W) + \dim(W') \\ W \cap W' = \{0_V\} \end{array} \right. \iff V = W \oplus W'$$

Proof: From Proposition 4.13, $W \cap W'$ is finite-dimensional as subspace of V. Let $\mathcal{B}_\cap = (v_1, \ldots, v_k)$ be a basis of $W \cap W'$. From the first point of Proposition 4.8, $\mathcal{B}_\cap$ is a subfamily of

- some basis of W, say $\mathcal{B} = (v_1, \ldots, v_k, w_1, \ldots, w_n)$
- some basis of W', say $\mathcal{B}' = (v_1, \ldots, v_k, w'_1, \ldots, w'_{n'})$

We are going to prove that $\mathcal{B}_+ = (v_1, \ldots, v_k, w_1, \ldots, w_n, w'_1, \ldots, w'_{n'})$ is a basis of $W + W'$.

Linearly independent. Assume $\lambda_1.v_1 + \cdots + \lambda_k.v_k + \mu_1.w_1 + \cdots + \mu_n.w_n + \mu'_1.w'_1 + \cdots + \mu'_{n'}.w'_{n'} = 0_V$ for some scalars $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n, \mu'_1, \ldots, \mu'_{n'}) \in \mathbb{K}^{k+n+n'}$. Then

$$w' = \mu'_1.w'_1 + \cdots + \mu'_{n'}.w'_{n'} = -(\lambda_1.v_1 + \cdots + \lambda_k.v_k + \mu_1.w_1 + \cdots + \mu_n.w_n) \in W \cap W'$$

Consequently $w' = \lambda'_1.v_1 + \cdots + \lambda'_k.v_k$ for some scalars $(\lambda'_1, \ldots, \lambda'_k) \in \mathbb{K}^k$. It follows

$$w' - w' = \lambda'_1.v_1 + \cdots + \lambda'_k.v_k + (-\mu'_1).w'_1 + \cdots + (-\mu'_{n'}).w'_{n'} = 0_V$$

Since $\mathcal{B}'$ is linearly independent, we get $\lambda'_1 = \cdots = \lambda'_k = \mu'_1 = \cdots = \mu'_{n'} = 0_\mathbb{K}$. Now we have $\lambda_1.v_1 + \cdots + \lambda_k.v_k + \mu_1.w_1 + \cdots + \mu_n.w_n = 0_V$ and thus $\lambda_1 = \cdots = \lambda_k = \mu_1 = \cdots = \mu_{n'} = 0_\mathbb{K}$ since $\mathcal{B}$ is linearly independent. Finally $\mathcal{B}_+$ is linearly independent.

Spanning family. Let v be a vector in $W + W'$. Then there exist $w \in W$ and $w' \in W'$ such that $v = w + w'$. Moreover

$$\left\{ \begin{array}{ll} \exists(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n) \in \mathbb{K}^{k+n} \ / & w = \lambda_1.v_1 + \cdots + \lambda_k.v_k + \mu_1.w_1 + \cdots + \mu_n.w_n \\ \exists(\lambda'_1, \ldots, \lambda'_k, \mu'_1, \ldots, \mu'_{n'}) \in \mathbb{K}^{k+n'} \ / & w' = \lambda'_1.v_1 + \cdots + \lambda'_k.v_k + \mu'_1.w'_1 + \cdots + \mu'_{n'}.w'_{n'} \end{array} \right.$$

It follows that

$$\begin{aligned} v &= w + w' \\ &= (\lambda_1.v_1 + \cdots + \lambda_k.v_k + \mu_1.w_1 + \cdots + \mu_n.w_n) + (\lambda'_1.v_1 + \cdots + \lambda'_k.v_k + \mu'_1.w'_1 + \cdots + \mu'_{n'}.w'_{n'}) \\ &= (\lambda_1 + \lambda'_1).v_1 + \cdots + (\lambda_k + \lambda'_k).v_k + \mu_1.w_1 + \cdots + \mu_n.w_n + \mu'_1.w'_1 + \cdots + \mu'_{n'}.w'_{n'} \end{aligned}$$

Consequently $\mathcal{B}_+$ is a spanning family of $W + W'$.

Finally $\mathcal{B}_+$ is a basis of $W + W'$. It follows that

$$\begin{aligned} \dim(W + W') &= \mathrm{Card}(\mathcal{B}_+) \\ &= k + n + n' \\ &= (k + n) + (k + n') - k \\ &= \mathrm{Card}(\mathcal{B}) + \mathrm{Card}(\mathcal{B}') - \mathrm{Card}(\mathcal{B}_\cap) \\ &= \dim(W) + \dim(W') - \dim(W \cap W') \end{aligned}$$

■

Remark: It follows from Proposition 4.8 that every subspace of a finite-dimensional $\mathbb{K}$-vector space has a supplementary subspace.

4.2.4 Linear map on finite-dimensional vector space

Proposition 4.15

Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces with V finite-dimensional. Then $\mathrm{Im}(f)$ is finite-dimensional.

Proof: Because if S is a spanning set of V then $f(S)$ is a spanning set of $\mathrm{Im}(f)$. ■

Definition 4.16 (*Rank*)

*Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces with V finite-dimensional. The **rank** of f, denoted **rank(f)**, is the dimension of the image of f.*

$$\mathrm{rank}(f) = \dim(\mathrm{Im}(f))$$

Theorem 4.17 (*Rank theorem*)

Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces with V finite-dimensional. Then

$$\dim(V) = \dim(\mathrm{Ker}(f)) + \mathrm{rank}(f)$$

Proof: Let $\mathcal{B}_{\mathrm{Ker}} = (v_1, \ldots, v_k)$ be a basis of $\mathrm{Ker}(f)$. From the first point of Proposition 4.8, $\mathcal{B}_{\mathrm{Ker}}$ is a subfamily of some basis of V, say $\mathcal{B} = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m})$. We are going to prove that $\mathcal{B}_{\mathrm{Im}} = (f(v_{k+1}), \ldots, f(v_{k+m}))$ is a basis of $\mathrm{Im}(f)$.

Linearly independent. Assume that $\lambda_{k+1}.f(v_{k+1}) + \cdots + \lambda_{k+m}.f(v_{k+m}) = 0_W$ for some scalars $(\lambda_{k+1}, \ldots, \lambda_{k+m}) \in \mathbb{K}^m$. Then $v = \lambda_{k+1}.v_{k+1} + \cdots + \lambda_{k+m}.v_{k+m}$ is in the kernel of f since

$$f(v) = \lambda_{k+1}.f(v_{k+1}) + \cdots + \lambda_{k+m}.f(v_{k+m}) = 0_W$$

Consequently $v = \lambda_1.v_1 + \cdots + \lambda_k.v_k$ for some scalars $(\lambda_1, \ldots, \lambda_k) \in \mathbb{K}^k$. It follows

$$v - v = \lambda_1.v_1 + \cdots + \lambda_k.v_k + (-\lambda_{k+1}).v_{k+1} + \cdots + (-\lambda_{k+m}).v_{k+m} = 0_V$$

Since $\mathcal{B}$ is linearly independent, we get $\lambda_1 = \cdots = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+m} = 0_{\mathbb{K}}$. In particular $\mathcal{B}_{\text{Im}}$ is linearly independent.

Spanning family. Let w be a vector in the image of f. Then there exists $v \in V$ such that $w = f(v)$. Moreover

$$\exists(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+m}) \in \mathbb{K}^{k+m} \;/\; v = \lambda_1.v_1 + \cdots + \lambda_k.v_k + \lambda_{k+1}.v_{k+1} + \cdots + \lambda_{k+m}.v_{k+m}$$

It follows that

$$\begin{aligned} w &= f(v) \\ &= f(\lambda_1.v_1 + \cdots + \lambda_k.v_k + \lambda_{k+1}.v_{k+1} + \cdots + \lambda_{k+m}.v_{k+m}) \\ &= f(\underbrace{\lambda_1.v_1 + \cdots + \lambda_k.v_k}_{\in \text{Ker}(f)}) + f(\lambda_{k+1}.v_{k+1} + \cdots + \lambda_{k+m}.v_{k+m}) \\ &= 0_W + f(\lambda_{k+1}.v_{k+1} + \cdots + \lambda_{k+m}.v_{k+m}) \\ &= \lambda_{k+1}.f(v_{k+1}) + \cdots + \lambda_{k+m}.f(v_{k+m}) \end{aligned}$$

Consequently $\mathcal{B}_{\text{Im}}$ is a spanning family of $\text{Im}(f)$.

Finally $\mathcal{B}_{\text{Im}}$ is a basis of $\text{Im}(f)$. It follows that

$$\begin{aligned} \dim(V) &= \text{Card}(\mathcal{B}) \\ &= k + m \\ &= \text{Card}(\mathcal{B}_{\text{Ker}}) + \text{Card}(\mathcal{B}_{\text{Im}}) \\ &= \dim(\text{Ker}(f)) + \text{rank}(f) \end{aligned}$$

■

Corollary 4.18

Let $f \in L(V)$ be a linear map from a finite-dimensional $\mathbb{K}$-vector space V to itself. Then the following properties are equivalent

1. *f is bijective (that is $f \in GL(V)$)*
2. *f is injective*
3. *f is surjective*
4. $\text{Ker}(f) = \{0_V\}$
5. $\text{rank}(f) = \dim(V)$

Proposition 4.19

Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces with V finite-dimensional. Then f is uniquely determined by the image of a basis of V. More precisely, if $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ is a basis of V then for every family $(w_1, w_2, \ldots, w_n)$ of vectors in W, there exists an unique linear map $f : V \to W$ such that $\forall k \in \{1, 2, \ldots, n\},\ f(v_k) = w_k$.

Proof : Since every vector $v \in V$ may be written uniquely as a linear combination of vectors in $\mathcal{B}$, the linear map f must be defined by

$$f(v) = \lambda_1.w_1 + \lambda_1.w_2 + \cdots + \lambda_1.w_n \quad \text{where } v = \lambda_1.v_1 + \lambda_1.v_2 + \cdots + \lambda_k.v_n$$

■

Proposition 4.20

Let V and W be two finite-dimensional $\mathbb{K}$-vector spaces. Then V and W are isomorphic if and only if they have the same dimension.

$$V \approx W \Longleftrightarrow \dim(V) = \dim(W)$$

In particular, every finite-dimensional $\mathbb{K}$-vector space V is isomorphic to $\mathbb{K}^{\dim(V)}$.

Proof : **Necessary.** Assume there exists a bijective linear map $f : V \to W$. Then $\mathrm{Ker}(f) = \{0_V\}$ since f is injective and $\dim(W) = \dim(\mathrm{Im}(V)) = \mathrm{rank}(f)$ since f is surjective. Rank theorem 4.17 gives $\dim(V) = 0 + \dim(W) = \dim(W)$.

Sufficient. Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be a basis of V and $\mathcal{B}' = (w_1, w_2, \ldots, w_n)$ be a basis of W. Now consider the linear map from V to W defined by

$$\forall k \in \{1, 2, \ldots, n\},\ f(v_k) = w_k$$

(f is well defined by Proposition 4.19.)
Then f is surjective since the image of f contains every vector of the spanning family $\mathcal{B}'$. In particular $\mathrm{rank}(f) = \dim(\mathrm{Im}(f)) = \dim(W) = \dim(V)$. Moreover Rank theorem 4.17 gives $\dim(\mathrm{Ker}(f)) = \dim(V) - \mathrm{rank}(f) = 0$ that is $\mathrm{Ker}(f) = \{0_V\}$. It follows that f is injective and thus an isomorphism. ■

Examples :

a) The dimension of $\mathbb{C}$ is 1 as a $\mathbb{C}$-vector space but 2 as a $\mathbb{R}$-vector space. More generally for any given integer $n \in \mathbb{N}$, the dimension of $\mathbb{C}^n$ is n as a $\mathbb{C}$-vector space but $2n$ as a $\mathbb{R}$-vector space.

b) For any given integer $d \in \mathbb{N}$, the space $\mathbb{K}_d[X]$ of all polynomials with degree at most d is isomorphic to the coordinate space $\mathbb{K}^{d+1}$ of dimension $d+1$ (for instance the map which associates to a polynomial the family of its first $d+1$ coefficients is an isomorphism).