# Chapter 3

# Vector spaces

In this chapter, fix a commutative field $(\mathbb{K}, +, \times)$ (for instance $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

## 3.1 The structure of vector space

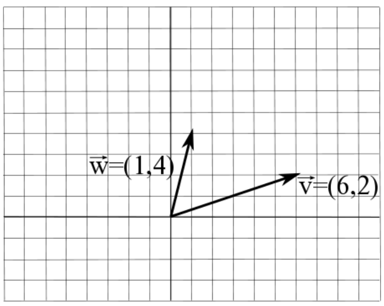
### 3.1.1 First examples

**Example of the real plane $\mathbb{R}^2$ :**

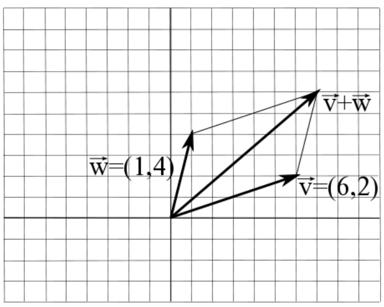
The set of all vectors in the plane consists of the set of all arrows starting at one fixed point in the plane . We may write it as follows

$$\mathbb{R}^2 = \{\overrightarrow{v} = (x, y) \ / \ x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$$

The real numbers $x$ and $y$ are called the **coordinates** of the vector $\overrightarrow{v}$.




Given two arrows $\overrightarrow{v}$ and $\overrightarrow{w}$ starting at one fixed point in a plane, the parallelogram spanned by these two arrows contains one diagonal arrow which starts at the same fixed point. This new arrow defines the sum of $\overrightarrow{v}$ and $\overrightarrow{w}$.

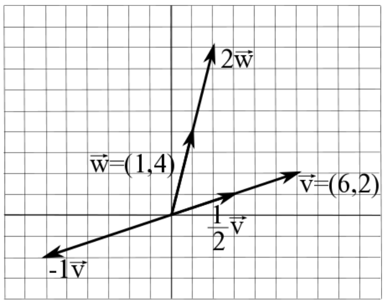


More precisely, the addition of two vectors in $\mathbb{R}^2$ is defined by the following binary operation

$$\begin{array}{rcl} +: \ \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (\overrightarrow{v}, \overrightarrow{w}) & \longmapsto & \overrightarrow{v} + \overrightarrow{w} = (x+x', y+y') \quad \text{where } \left\{ \begin{array}{l} \overrightarrow{v} = (x,y) \\ \overrightarrow{w} = (x',y') \end{array} \right. \end{array}$$

Notice that this binary operation provides an abelian group structure on $\mathbb{R}^2$.

Moreover, any arrow $\overrightarrow{v}$ starting at one fixed point in a plane may be scaled: given any positive real number $\lambda$, the scaling of $\overrightarrow{v}$ by $\lambda$ is the arrow whose diection is the same as $\overrightarrow{v}$ but is dilated or shrunk by multiplying its length by $\lambda$. When $\lambda$ is negative, the scaling of $\overrightarrow{v}$ by $\lambda$ is defined as the arrow pointing in the opposite direction, instead.




More precisely, the multiplication of a vector in $\mathbb{R}^2$ by a scalar is defined as follows

$$\begin{array}{rcl} .: \ \mathbb{R} \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (\lambda, \overrightarrow{v}) & \longmapsto & \lambda.\overrightarrow{v} = (\lambda x, \lambda y) \quad \text{where } \overrightarrow{v} = (x,y) \end{array}$$

But notice that . is not a binary operation.

**Example of the real space $\mathbb{R}^3$ :**

Similarly to the plane, it is the same for the space. Indeed the set of all vectors in the space consists of the set of all arrows starting at one fixed point in the space.

$$\mathbb{R}^3 = \{\overrightarrow{v} = (x,y,z) \ / \ x \in \mathbb{R}, \ y \in \mathbb{R} \text{ and } z \in \mathbb{R}\}$$

Each vector in the space $\mathbb{R}^3$ has three coordinates (instead of two for a vector in the plane $\mathbb{R}^2$).

The addition of two vectors in $\mathbb{R}^3$ is defined by the following binary operation

$$\begin{array}{rcl} +: \ \mathbb{R}^3 \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (\overrightarrow{v}, \overrightarrow{w}) & \longmapsto & \overrightarrow{v} + \overrightarrow{w} = (x+x', y+y', z+z') \quad \text{where } \left\{ \begin{array}{l} \overrightarrow{v} = (x,y,z) \\ \overrightarrow{w} = (x',y',z') \end{array} \right. \end{array}$$

The multiplication of a vector in $\mathbb{R}^3$ by a scalar is defined as follows

$$\begin{array}{rcl} .: \ \mathbb{R} \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (\lambda, \overrightarrow{v}) & \longmapsto & \lambda.\overrightarrow{v} = (\lambda x, \lambda y, \lambda z) \quad \text{where } \overrightarrow{v} = (x,y,z) \end{array}$$

We get a similar algebraic structure as for the real plane $\mathbb{R}^2$: a binary operation + such that $(\mathbb{R}^3, +)$ is an abelian group and a multiplication by a scalar which is not a binary operation.

### 3.1.2 Definitions

**Definition 3.1 (*External binary operation*)**

*An **external binary operation** (or external binary law) over $\mathbb{K}$ on a nonempty set $S$ is a map from $\mathbb{K} \times S$ to $S$. Such a binary operation is usually denoted*

$$\begin{array}{rccc} .: & \mathbb{K} \times S & \longrightarrow & S \\ & (\lambda, x) & \longmapsto & \lambda.x \end{array}$$

**Definition 3.2 (*Vector space*)**

*Let $(V, +, .)$ be a nonempty set with a binary operation denoted $+$ and an external binary operation over $\mathbb{K}$ denoted $.$ and called the **scalar multiplication**. $(V, +, .)$ (or simply $V$) is said to be a **vector space over** $\mathbb{K}$ (or a $\mathbb{K}$**-vector space**) if it satisfies each of the following vector space axioms*

**i)** *$(V, +)$ is an abelian group*

**ii)** *the scalar multiplication $.$ is **distributive** over the binary operation $+$ on $V$:*

$$\forall \lambda \in \mathbb{K},\ \forall (v, w) \in V^2,\ \lambda.(v + w) = \lambda.v + \lambda.w$$

**iii)** *the scalar multiplication $.$ is **distributive** over the binary operation $+$ on $\mathbb{K}$:*

$$\forall (\lambda, \mu) \in \mathbb{K}^2,\ \forall v \in V,\ (\lambda + \mu).v = \lambda.v + \mu.v$$

**iv)** *the scalar multiplication $.$ is **compatible** with the binary operation $\times$ on $\mathbb{K}$:*

$$\forall (\lambda, \mu) \in \mathbb{K}^2,\ \forall v \in V,\ (\lambda \times \mu).v = \lambda.(\mu.v)$$

**v)** *$1_{\mathbb{K}} \in \mathbb{K}$ is the **identity element** for the scalar multiplication $.$:*

$$\forall v \in V,\ 1_{\mathbb{K}}.v = v$$

*In this case, the elements of $\mathbb{K}$ are called **scalars** and those of $V$ are called **vectors**. Furthermore if $\lambda_1, \lambda_2, \dots \lambda_n$ are scalars and $v_1, v_2, \dots, v_n$ are vectors, then the **linear combination** of those vectors with those scalars is given by*

$$\lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n = \sum_{k=1}^{n} \lambda_k.v_k \in V$$

**Examples :** The real plane $\mathbb{R}^2$ and the real space $\mathbb{R}^3$ are vector spaces over $\mathbb{R}$.

**Proof :** Notice that the scalar multiplication $.$ on $\mathbb{R}^2$ or $\mathbb{R}^3$ corresponds for every coordinate to the multiplication $\times$ on $\mathbb{R}$. Consequently, the distributivity of the scalar multiplication $.$ over the addition $+$ on $V$ or $\mathbb{R}$ follows from the distributivity of the multiplication $\times$ over the addition $+$ on $\mathbb{R}$, the compatibility of the scalar multiplication $.$ with the multiplication $\times$ on $\mathbb{R}$ follows from the associativity of the multiplication $\times$ on $\mathbb{R}$, and $1_{\mathbb{R}} \in \mathbb{R}$ is the identity element for the scalar multiplication $.$ since it is the identity element for the multiplication $\times$ on $\mathbb{R}$.

■

**Proposition 3.3**

*Let $(V,+,.)$ be a vector space over $\mathbb{K}$. Then the following properties hold*

1. $\forall \lambda \in \mathbb{K},\ \lambda.0_V = 0_V$
2. $\forall v \in V,\ 0_{\mathbb{K}}.v = 0_V$
3. $\forall \lambda \in \mathbb{K},\ \forall v \in V,\ \lambda.v = 0_V \Longrightarrow$ *either* $\lambda = 0_{\mathbb{K}}$ *or* $v = 0_V$
4. $\forall \lambda \in \mathbb{K},\ \forall v \in V,\ (-\lambda).v = -(\lambda.v)$

**Proof:**

1. From the distributivity of the scalar multiplication over $+$ on $V$ we get

$$\lambda.0_V + \lambda.0_V = \lambda.(0_V + 0_V) = \lambda.0_V = 0_V + \lambda.0_V$$

Now, a simplification on each side by $\lambda.0_V$ gives $\lambda.0_V = 0_V$ as needed.

2. From the distributivity of the scalar multiplication over $+$ on $\mathbb{K}$ we get

$$0_{\mathbb{K}}.v + 0_{\mathbb{K}}.v = (0_{\mathbb{K}} + 0_{\mathbb{K}}).v = 0_{\mathbb{K}}.v = 0_V + 0_{\mathbb{K}}.v$$

Now, a simplification on each side by $0_{\mathbb{K}}.v$ gives $0_{\mathbb{K}}.v = 0_V$ as needed.

3. Assume $\lambda \neq 0_{\mathbb{K}}$. Consequently, there exists a multiplicative inverse $\lambda^{-1} \in \mathbb{K}$ and using the compatibility of the scalar multiplication with $\times$ on $\mathbb{K}$ and the first point, we get

$$v = 1_{\mathbb{K}}.v = (\lambda^{-1} \times \lambda).v = \lambda^{-1}.(\lambda.v) = \lambda^{-1}.0_V = 0_V$$

Finally either $\lambda = 0_{\mathbb{K}}$ or $v = 0_V$.

4. From the distributivity of the scalar multiplication over $+$ on $\mathbb{K}$ and the second point we get

$$\lambda.v + (-\lambda.v) = (\lambda + (-\lambda)).v = 0_{\mathbb{K}}.v = 0_V$$

■

### 3.1.3 More examples

**Definition 3.4 (*Coordinate space*)**

*Fix a positive integer $n \in \mathbb{N}^*$. The* ***coordinate space*** *of dimension $n$ is the set of all $n$-tuples of elements of $\mathbb{K}$ which is denoted*

$$\mathbb{K}^n = \{v = (x_1, x_2, \ldots, x_n)\ /\ x_1 \in \mathbb{K}, x_2 \in \mathbb{K}, \ldots, \text{ and } x_n \in \mathbb{K}\}$$

*It is a $\mathbb{K}$-vector space for the following operations*

$$\begin{array}{rcl} + : \mathbb{K}^n \times \mathbb{K}^n & \longrightarrow & \mathbb{K}^n \\ (v,w) & \longmapsto & v + w = (x_1 + x'_1, x_2 + x'_2, \ldots, x_n + x'_n) \quad \text{where } \left\{ \begin{array}{l} v = (x_1, x_2, \ldots, x_n) \\ w = (x'_1, x'_2, \ldots, x'_n) \end{array} \right. \end{array}$$

$$\begin{array}{rcl} . : \mathbb{K} \times \mathbb{K}^n & \longrightarrow & \mathbb{K}^n \\ (\lambda, v) & \longmapsto & \lambda.v = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \quad \text{where } v = (x_1, x_2, \ldots, x_n) \end{array}$$

*The elements $x_1, x_2, \ldots, x_n$ in $\mathbb{K}$ are called the* ***coordinates*** *of the vector $v = (x_1, x_2, \ldots, x_n)$.*

**Examples :**

**a)** For $\mathbb{K} = \mathbb{R}$, we get the $\mathbb{R}$-vector space $(\mathbb{R}^n, +, .)$ called the **real coordinate space** (or simply the **real plane** in case $n = 2$ and the **real space** in case $n = 3$). And for $\mathbb{K} = \mathbb{C}$, we get the $\mathbb{C}$-vector space $(\mathbb{C}^n, +, .)$ called the **complex coordinate space**.

**b)** In particular $n = 1$ and $\mathbb{K} = \mathbb{R}$ gives that $\mathbb{R}$ is a $\mathbb{R}$-vector space. In this case, the scalar multiplication $.$ corresponds to the multiplication $\times$ on $\mathbb{R}$ (and then the external binary operation is actually a binary operation).

**c)** Similarly, $\mathbb{C}$ is a $\mathbb{C}$-vector space.

**d)** But $\mathbb{C}$ is also a $\mathbb{R}$-vector space. To show this, we may identify $\mathbb{C}$ and its operations as follows

$$\mathbb{C} = \{z = a + \imath b \ / \ a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$$

$$\begin{array}{rrcl} + : & \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (z, z') & \longmapsto & z + z' = (a + a') + \imath(b + b') \quad \text{where } \left\{ \begin{array}{l} z = a + \imath b \\ z' = a' + \imath b' \end{array} \right. \end{array}$$

$$\begin{array}{rrcl} . : & \mathbb{R} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (\lambda, z) & \longmapsto & \lambda.z = (\lambda a) + \imath(\lambda b) \quad \text{where } z = a + \imath b \end{array}$$

Actually, $\mathbb{C}$ as a $\mathbb{R}$-vector space is very "similar" to the real plane $\mathbb{R}^2$ (the coordinates of a vector $z = a + \imath b \in \mathbb{C}$ are $a \in \mathbb{R}$ and $b \in \mathbb{R}$).

**Further examples :**

**a) Polynomial space.** The ring $\mathbb{K}[X]$ of all polynomials with coefficients in $\mathbb{K}$ is a $\mathbb{K}$-vector space. The addition comes from the ring structure and the scalar multiplication is defined as follows

$$\begin{array}{rrcl} . : & \mathbb{K} \times \mathbb{K}[X] & \longrightarrow & \mathbb{K}[X] \\ & (\lambda, P(X)) & \longmapsto & (\lambda.P)(X) = \lambda P(X) = \displaystyle\sum_{n=0}^{+\infty} \lambda a_n X^n \quad \text{where } P(X) = \displaystyle\sum_{n=0}^{+\infty} a_n X^n \end{array}$$

**b) Function space.** The set $\mathcal{F}(S, V) = \{f : S \to V, x \mapsto f(x)\}$ of all functions from a set $S$ to a $\mathbb{K}$-vector space $(V, +, .)$ is a $\mathbb{K}$-vector space with the following operations

$$\begin{array}{rrcl} + : & \mathcal{F}(S, V) \times \mathcal{F}(S, V) & \longrightarrow & \mathcal{F}(S, V) \\ & (f, g) & \longmapsto & (f + g) : x \mapsto f(x) + g(x) \end{array}$$

$$\begin{array}{rrcl} . : & \mathbb{K} \times \mathcal{F}(S, V) & \longrightarrow & \mathcal{F}(S, V) \\ & (\lambda, f) & \longmapsto & (\lambda.f) : x \mapsto \lambda.f(x) \end{array}$$

In particular, the set $\mathcal{F}(\mathbb{K}) = \mathcal{F}(\mathbb{K}, \mathbb{K}) = \{f : \mathbb{K} \to \mathbb{K}, x \mapsto f(x)\}$ of all functions from $\mathbb{K}$ to itself is a $\mathbb{K}$-vector space.

**c) Sequence space.** Choosing $S$ in the previous example to be the set of all natural numbers $\mathbb{N}$, we get the $\mathbb{K}$-vector space $\mathcal{F}(\mathbb{N}, V) = \{u : \mathbb{N} \to V, n \mapsto u_n\}$ of all infinite sequences of elements of $V$.

## 3.2 Subspace

### 3.2.1 Definition and characterization

**Definition 3.5** (***Subspace***)

*Let $(V, +, .)$ be a $\mathbb{K}$-vector space and $W \subset V$ be a nonempty subset. $(W, +, .)$ (or simply $W$) is a* ***subspace*** *of $V$ over $\mathbb{K}$ if it satisfies each of the following subspace axioms*

**i)** *$(W, +)$ is a subgroup of $(V, +)$*

**ii)** *. is an external binary operation over $\mathbb{K}$ on $W$*

*In this case, $(W, +, .)$ is a $\mathbb{K}$-vector space.*

**Trivial example :** For any $\mathbb{K}$-vector space $(V, +, .)$, $(\{0_V\}, +, .)$ is a subspace called the **trivial subspace**.

**Proposition 3.6**

*Let $(V, +, .)$ be a $\mathbb{K}$-vector space and $W \subset V$ be a nonempty subset. $W$ is a subspace of $V$ over $\mathbb{K}$ if and only if it satisfies the following condition*

$$\forall \lambda \in \mathbb{K},\ \forall (v, w) \in W^2,\ v + \lambda.w \in W$$

**Proof :** **Necessary.** If $(W, +, .)$ is a $\mathbb{K}$-vector space then $v + \lambda.w = 1_{\mathbb{K}}.v + \lambda.w \in W$ as a linear combination of vectors in $W$.

**Sufficient.** $\lambda = -1_{\mathbb{K}}$ gives $v + (-w) \in W$ for every vectors $v \in W$ and $w \in W$. Consequently $(W, +)$ is a subroup of $(V, +)$. In particular $0_V$ is in $W$ and $v = 0_V$ gives $\lambda.w \in W$ for every $\lambda \in \mathbb{K}$ and $w \in W$ that is . is an external binary operation over $\mathbb{K}$ on $W$. The conclusion follows. ■

**Remark :** Equivalently, a nonempty subset $W$ of a vector space $(V, +, .)$ is a subspace if and only if any linear combination of vectors in $W$ belongs to $W$.

**Some examples :**

**a)** Let $v$ be a non zero vector in a $\mathbb{K}$-vector space $(V, +, .)$, that is $v \in V - \{0_V\}$. Consider the following set of all vectors **colinear** to $v$

$$\mathbb{K}.v = \{\lambda.v \ / \ \lambda \in \mathbb{K}\}$$

Then $\mathbb{K}.v$ is a subspace of $V$ called the **linear line spanned by $\boldsymbol{v}$**.

**Proof :** For every scalar $\mu \in \mathbb{K}$ and every vectors $w = \lambda.v$ and $w' = \lambda'.v$ in $\mathbb{K}.v$, we have:

$$w + \mu.w' = (\lambda.v) + \mu.(\lambda'.v) = (\lambda + \mu\lambda').v \in \mathbb{K}.v$$

The conclusion follows from Proposition 3.6. ■

**b)** Let $a$, $b$ and $c$ be three real numbers. Consider the following subset of the real space $\mathbb{R}^3$

$$\mathcal{P} = \big\{v = (x, y, z) \in \mathbb{R}^3 \ / \ ax + by + cz = 0\big\}$$

Then $\mathcal{P}$ is a subspace of $\mathbb{R}^3$.

**Proof:** For every real $\lambda \in \mathbb{R}$ and every vectors $v = (x, y, z)$ and $w = (x, y, z)$ in $\mathcal{P}$ we have $v + \lambda.w = (x + \lambda x', y + \lambda y', z + \lambda z')$ and

$$\begin{aligned} a(x+\lambda x') + b(y+\lambda y') + c(z+\lambda z') &= (ax+by+cz) + \lambda(ax'+by'+cz') \\ &= 0 + \lambda \times 0 \\ &= 0 \end{aligned}$$

Then $v + \lambda.w$ is in $\mathcal{P}$ and the conclusion follows from Proposition 3.6. ■

**c) Counterexample.** Consider the following subset of the real plane $\mathbb{R}^2$

$$\mathcal{C} = \left\{ v = (x, y) \in \mathbb{R}^2 \ / \ x^2 + y^2 = 1 \right\}$$

Then $\mathcal{C}$ is not a subspace of $\mathbb{R}^2$. Indeed notice that $v = (1, 0)$ and $w = (0, 1)$ are in $\mathcal{C}$ but their linear combination $v + w = (1, 1)$ is not. Furthermore, the zero vector $0_{\mathbb{R}^2} = (0, 0)$ is not in $\mathcal{C}$.

**Further examples:**

**a) Polynomial subspaces.** Fix $d \in \mathbb{N}$. The set $\mathbb{K}_d[X]$ of all polynomials with degree at most $d$ is a subspace of $\mathbb{K}[X]$ since for every scalar $\lambda \in \mathbb{K}$ and every polynomials $P(X)$ and $Q(X)$ in $\mathbb{K}_d[X]$

$$\deg(P + \lambda.Q) \leqslant \max\{\deg(P), \deg(\lambda.Q)\} \leqslant \max\{\deg(P), \deg(Q)\} \leqslant d$$

But the set of all polynomials with degree exactly $d$ is not (for instance the linear combination $X^d + (-1_{\mathbb{K}}).X^d = 0_{\mathbb{K}}$ of polynomials of degree $d$ is not of degree $d$).

**b) Function subspaces.** Let $S$ be a nonempty set, $a$ be an element in $S$ and $(V, +, .)$ be a $\mathbb{K}$-vector space. Then the set of all functions $f : S \to V$ such that $f(a) = 0_V$ is a subspace of $\mathcal{F}(S, V)$. But the set of all functions $f : S \to V$ such that $f(a)$ is a given non zero vector in $V$ is not.

Furthermore, the set of all continuous (respectively differentiable) functions from $\mathbb{R}$ to itself is a subspace of $\mathcal{F}(\mathbb{R})$ since the sum and the product of any continuous (respectively differentiable) functions is continuous (respectively differentiable).

**c) Sequence subspaces.** The set of all infinite sequences of elements in a $\mathbb{K}$-vector space $(V, +, .)$ which are eventually equal to zero (that is all elements except a finite number are equal to $0_V$) is a subspace of $\mathcal{F}(\mathbb{N}, V)$. But the set of all infinite sequences which are eventually equal to a given non zero vector is not.

Furthermore, the set of all infinite sequences of elements in $\mathbb{K}$ which are convergent is a subspace of $\mathcal{F}(\mathbb{N}, \mathbb{K})$. The same goes for the set of all infinite sequences of elements in $\mathbb{K}$ which converge to $0_{\mathbb{K}}$. But the set of all infinite sequences of elements in $\mathbb{K}$ which converge to a given non zero element is not.

### Proposition 3.7

*Let $W$ and $W'$ be two subspaces of a $\mathbb{K}$-vector space $(V, +, .)$. Then $W \cap W'$ is also a subspace of $V$.*

**Proof:** The result follows from Proposition 3.6 ■

**Remark:** The same does not hold for $W \cup W'$ as soon as $W \not\subset W'$ and $W' \not\subset W$.

**Proof:** Let $w$ and $w'$ be two vectors in $W \cup W'$ such that $w \in W - W'$ and $w' \in W' - W$. Then the vector $v = w + w'$ is neither in $W$ (otherwise $w' = v - w$ would be in $W$ as a linear combination of vectors in $W$ that is a contradiction since $w' \notin W$) nor in $W$ (by a similar contradiction for $w = v - w'$). So $W \cup W'$ does not contain the linear combination $w + w'$ of vectors in $W \cup W'$ and consequently $W \cup W'$ is not a subspace (from Proposition 3.6). ■

### 3.2.2 Linear span

**Definition 3.8**

*Let $(V,+,.)$ be a $\mathbb{K}$-vector space and $S \subset V$ be a subset of vectors. The* ***linear span*** *of $S$, denoted* **Span($S$)***, is the smallest subspace of $V$ over $\mathbb{K}$ containing $S$.*

**Remark:** More precisely, Span($S$) is a subspace containing $S$ and such that Span($S$) $\subset W$ for every subspace $W$ containing $S$. Hence, the linear span of $S$ is given by the following intersection

$$\mathrm{Span}(S) = \bigcap_{\substack{W \text{ subspace of } (V,+,.) \\ \text{such that } S \subset W}} W$$

**Proof:** Denote by $I(S)$ the intersection above. It follows from Proposition 3.7 that $I(S)$ is a subspace. Moreover $I(S)$ contains $S$ since $S$ is included in every set of the intersection. Then Span($S$) $\subset I(S)$ by definition of the linear span of $S$.

Conversely, Span($S$) is a subspace containing $S$. It follows that Span($S$) is one of the sets of the intersection $I(S)$ and then $I(S) \subset$ Span($S$). Finally, Span($S$) $= I(S)$ as needed. ■

**Proposition 3.9**

*Let $(V,+,.)$ be a $\mathbb{K}$-vector space and $S \subset V$ be a subset of vectors. Then the linear span of $S$ is the set of all linear combinations of vectors in $S$.*

$$\mathrm{Span}(S) = \left\{ \lambda_1.v_1 + \lambda_2.v_2 + \cdots + \lambda_n.v_n \;/\; n \in \mathbb{N} \text{ and } \left\{ \begin{array}{l} (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^n \\ (v_1, v_2, \ldots, v_n) \in S^n \end{array} \right. \right\}$$

**Proof:** Denote by $L(S)$ the set of all linear combinations of vectors in $S$. $L(S)$ is a subspace containing $S$ (with linear combination of the form $n = 1$ and $\lambda_1 = 1_{\mathbb{K}}$). Consequently Span($S$) $\subset L(S)$ by definition of the linear span of $S$.

Conversely, any linear combination of vectors in $S$ is a linear combination of vectors in the subspace Span($S$) (since $S \subset$ Span($S$)) and then is in Span($S$) (from Proposition 3.6). It follows that $L(S) \subset$ Span($S$) and finally Span($S$) $= L(S)$ as needed. ■

**Examples:**

**a)** Span($\emptyset$) $= \{0_V\}$

**b)** For every subspace $W$, we have Span($W$) $= W$.

**c)** The linear line spanned by a non zero vector $v$ is defined to be

$$\mathbb{K}.v = \mathrm{Span}(\{v\}) = \{\lambda.v \;/\; \lambda \in \mathbb{K}\}$$

**d)** Consider the following subspace of the real space $\mathbb{R}^3$

$$\mathcal{P} = \left\{ v = (x,y,z) \in \mathbb{R}^3 \;/\; x - 2y + z = 0 \right\}$$

Then $\mathcal{P} =$ Span($S$) where $S = \{(2,1,0), (-1,0,1)\}$.

**Proof:** At first $(2,1,0)$ and $(-1,0,1)$ are in $\mathcal{P}$ since $2 - 2 \times 1 + 0 = 0$ and $-1 - 2 \times 0 + 1 = 0$. Consequently, Span($S$) $\subset \mathcal{P}$. Converserly, let $v = (x,y,z)$ be a vector in $\mathcal{P}$. Then we have

$$x = 2y - z \text{ and } v = (x,y,z) = (2y - z, y, z) = y.(2,1,0) + z.(-1,0,1)$$

It follows that $v \in$ Span($S$) as a linear combination of $(2,1,0)$ and $(-1,0,1)$, then $\mathcal{P} \subset$ Span($S$). ■

**Further examples :**

**a) Polynomial space.** For any element $a \in \mathbb{K}$, the exact Taylor's formula implies that the vector space $\mathbb{K}[X]$ is spanned by the infinite set $\{(X-a)^k \ / \ k \in \mathbb{N}\}$. Similarly given $d \in \mathbb{N}$, the subspace $\mathbb{K}_d[X]$ of all polynomials with degree at most $d$ is spanned by the finite set $\{(X-a)^k \ / \ k \in \{0, 1, \dots, d\}\}$, for any element $a \in \mathbb{K}$.

**b) Function space.** The linear span of $\{f_k : \mathbb{K} \to \mathbb{K}, x \mapsto x^k \ / \ k \in \mathbb{N}\}$ is the subspace of $\mathcal{F}(\mathbb{K})$ of all polynomial maps.

**c) Sequence subspaces.** For every integer $k \in \mathbb{N}$, define an infinite sequence $u^k \in \mathcal{F}(\mathbb{N}, \mathbb{K})$ as follows

$$\forall i \in \mathbb{N},\ u_i^k = \begin{cases} 1_{\mathbb{K}} & \text{if } i = k \\ 0_{\mathbb{K}} & \text{otherwise} \end{cases}$$

Then the linear span of $S = \{u^k \ / \ k \in \mathbb{N}\}$ is not the whole vector space $\mathcal{F}(\mathbb{N}, \mathbb{K})$. For instance, the infinite sequence $v \in \mathcal{F}(\mathbb{N}, \mathbb{K})$ whose all elements are equal to $1_{\mathbb{K}}$ can not be written as a linear combination of vectors in $S$.

**Proof :** By contradiction, assume that $v$ can be written as a linear combination of vectors in $S$. Then there exist some integers $k_1, k_2, \dots, k_n$ such that

$$v = \lambda_1.u^{k_1} + \lambda_2.u^{k_2} + \cdots + \lambda_n.u^{k_n}$$

Choose an integer $i$ distinct from $k_1$, $k_2$, $\dots$, and $k_n$. Then

$$\begin{aligned} & u_i^{k_1} = u_i^{k_2} = \cdots = u_i^{k_n} = 0_{\mathbb{K}} \\ \text{and} \quad & (\lambda_1.u^{k_1} + \lambda_2.u^{k_2} + \cdots + \lambda_n.u^{k_n})_i = \lambda_1 u_i^{k_1} + \lambda_2 u_i^{k_2} + \cdots + \lambda_n u_i^{k_n} = 0_{\mathbb{K}} \end{aligned}$$

That is a contradiction with $v_i = 1_{\mathbb{K}}$. ■

Actually, Span$(S)$ is the subspace of all infinite sequences of elements in $\mathbb{K}$ which are eventually equal to zero.

### 3.2.3 Sum and direct sum

**Definition 3.10 (*Sum of subspaces*)**

*Let $W$ and $W'$ be two subspaces of a $\mathbb{K}$-vector space $(V, +, .)$. The **sum** of $W$ and $W'$ is defined to be the following set*

$$W + W' = \{w + w' \ / \ w \in W \text{ and } w' \in W'\}$$

**Proposition 3.11**

*Let $W$ and $W'$ be two subspaces of a $\mathbb{K}$-vector space $(V, +, .)$. Then*

$$W + W' = \text{Span}(W \cup W')$$

*In particular, $W + W'$ is a subspace of $V$.*

**Proof :** Let $v$ be a vector in $W + W'$. Then there exist $w \in W$ and $w' \in W'$ such that $v = w + w'$. In other words, $v$ is a linear combination of vectors in $W \cup W'$. Consequently $v \in \text{Span}(W \cup W')$ and then $W + W' \subset \text{Span}(W \cup W')$.

Conversely, let $v$ be a vector in $\mathrm{Span}(W \cup W')$. Then $v$ is a linear combination of vectors in $W \cup W'$ that is

$$\exists n \in \mathbb{N}/ \left\{ \begin{array}{l} \exists(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^n \\ \exists(w_1, w_2, \ldots, w_n) \in (W \cup W')^n \end{array} \right. / \; v = \lambda_1.w_1 + \lambda_2.w_2 + \cdots + \lambda_n.w_n$$

Without loss of generality, we may reorder the vectors $w_1$, $w_2$, ..., and $w_n$ such that the first ones are in $W$ and the next ones are in $W'$. More precisely, we may assume there exists $k \in \{0, 1, \ldots, n\}$ such that

$$(w_1, w_2, \ldots, w_k) \in W^k \quad \text{and} \quad (w_{k+1}, w_{k+2}, \ldots, w_n) \in (W')^k$$

Then we may write $v$ as follows

$$v = \underbrace{\lambda_1.w_1 + \lambda_2.w_2 + \cdots + \lambda_k.w_k}_{w} + \underbrace{\lambda_{k+1}.w_{k+1} + \lambda_{k+2}.w_{k+2} + \cdots + \lambda_n.w_n}_{w'} = w + w'$$

with $\left\{ \begin{array}{l} w = \lambda_1.w_1 + \lambda_2.w_2 + \cdots + \lambda_k.w_k \in W \text{ as a linear combination of vectors in W} \\ w' = \lambda_{k+1}.w_{k+1} + \lambda_{k+2}.w_{k+2} + \cdots + \lambda_n.w_n \in W' \text{ as a linear combination of vectors in W'} \end{array} \right.$

Consequently $v = w + w' \in W + W'$ and $\mathrm{Span}(W \cup W') \subset W + W'$. ∎

**Example:** Let $\mathbb{K}.v$ and $\mathbb{K}.v'$ be two linear lines spanned by two non zero vectors $v$ and $v'$. Then

$$\mathbb{K}.v + \mathbb{K}.v' = \left\{ \lambda.v + \lambda'.v' \; / \; (\lambda, \lambda') \in \mathbb{K}^2 \right\} = \mathrm{Span}(\{v, v'\})$$

In particular, the sum of the linear lines $\mathbb{K}.v$ and $\mathbb{K}.v'$ is a linear line if and only if $v$ and $v'$ are colinear (that is there exists $\mu \in \mathbb{K}^*$ such that $v = \mu.v'$). In this case $\mathbb{K}.v = \mathbb{K}.v' = \mathbb{K}.v + \mathbb{K}.v'$.

**Proof:** At first, assume $\mathbb{K}.v + \mathbb{K}.v' = \mathbb{K}.w$ for some non zero vector $w$. We have:

$$\left\{ \begin{array}{l} v = 1_{\mathbb{K}}.v + 0_{\mathbb{K}}.v' \in \mathbb{K}.v + \mathbb{K}.v' = \mathbb{K}.w \\ v' = 0_{\mathbb{K}}.v + 1_{\mathbb{K}}.v' \in \mathbb{K}.v + \mathbb{K}.v' = \mathbb{K}.w \end{array} \right. \quad \text{then} \quad \left\{ \begin{array}{l} \exists\lambda \in \mathbb{K}/ \; v = \lambda.w \\ \exists\lambda' \in \mathbb{K}/ \; v' = \lambda'.w \end{array} \right.$$

Moreover $\lambda$ and $\lambda'$ are not equal to $0_{\mathbb{K}}$ since $v$ and $v'$ are not equal to $0_V$. It follows that $w = \lambda'^{-1}.v'$ and $v = \lambda.w = (\lambda\lambda'^{-1}).v' = \mu.v'$ with $\mu = \lambda\lambda'^{-1} \in \mathbb{K}^*$.
Conversly, assume $v = \mu.v'$ for some $\mu \in \mathbb{K}^*$. Then

$$\mathbb{K}.v + \mathbb{K}.v' = \left\{ \lambda.v + \lambda'.(\mu.v) \; / \; (\lambda, \lambda') \in \mathbb{K}^2 \right\} = \left\{ (\lambda + \lambda'\mu).v \; / \; (\lambda, \lambda') \in \mathbb{K}^2 \right\} = \{ \lambda.v \; / \; \lambda \in \mathbb{K} \} = \mathbb{K}.v$$

In particular, $\mathbb{K}.v + \mathbb{K}.v'$ is a linear line. Furthermore, a similar reasoning shows $\mathbb{K}.v + \mathbb{K}.v' = \mathbb{K}.v'$ (with $v' = \mu^{-1}.v$). The conclusion follows. ∎

**Remarks:**

- More generally, if $S$ and $S'$ are two subsets of vectors then

$$\mathrm{Span}(S) + \mathrm{Span}(S') = \mathrm{Span}(S \cup S')$$

- That provides a binary operation $+$ on the set $\mathcal{V}$ of all subspaces of a $\mathbb{K}$-vector space $(V, +, .)$. This binary operation is associative and commutative. Moreover the trivial subspace $\{0_V\}$ is an identity element. But $(\mathcal{V}, +)$ is not an abelian group since any non trivial subspace has no inverse element. Actually, the same holds for the binary operation $\cap$ on $\mathcal{V}$ (with the whole subspace $V$ as identity element).

- The sum $W + W'$ of two subspaces of a $\mathbb{K}$-vector space $(V, +, .)$ satisfies the following property

$$\forall v \in W + W',\ \exists (w, w') \in W \times W' /\ v = w + w'$$

But the vectors $w$ and $w'$ are not necessarily unique. For instance if $W = W' \neq \emptyset$ then $W + W = W$ and the zero vector $0_V \in W$ may be written as $0_V = w + (-w)$ for every vector $w \in W$.

**Definition 3.12 (*Direct sum of subspaces*)**

*Let $W$ and $W'$ be two subspaces of a $\mathbb{K}$-vector space $(V, +, .)$. The sum $W + W'$ is said to be* ***direct*** *if every vector in $W + W'$ may be written uniquely as a sum $w + w'$ with $w \in W$ and $w' \in W'$. Equivalently, $W + W'$ is a direct sum if it satisfies the following condition*

$$\forall v \in W + W',\ \exists ! (w, w') \in W \times W' /\ v = w + w'$$

*In this case we write the sum $W \oplus W'$.*

**Proposition 3.13**

*Let $W$ and $W'$ be two subspaces of a $\mathbb{K}$-vector space $(V, +, .)$. Then the sum $\Sigma = W + W'$ is a direct sum if and only if $W$ and $W'$ have no intersection except the zero vector. Equivalently*

$$\Sigma = W \oplus W' \Longleftrightarrow \begin{cases} \Sigma = W + W' \\ W \cap W' = \{0_V\} \end{cases}$$

**Proof:** **Necessary.** Let $v$ be a vector in $W \cap W'$. Then we may write

$$\begin{aligned} v &= v + 0_V \quad \text{with } v \in W \text{ and } 0_V \in W' \\ &= 0_V + v \quad \text{with } 0_V \in W \text{ and } v \in W' \end{aligned}$$

In particular we get that $v$ is in $W + W' = \Sigma = W \oplus W'$ and then $v$ may be written uniquely as a sum of a vector in $W$ and a vector in $W'$. It follows that $v = 0_V$ and finally $W \cap W' = \{0_V\}$.

**Sufficient.** Let $v$ be a vector in $\Sigma = W + W'$. By contradiction, assume there exist two vectors $w_1$ and $w_2$ in $W$ and two vectors $w'_1$ and $w'_2$ in $W'$ such that $v = w_1 + w'_1 = w_2 + w'_2$. Then the vector $w = w_1 - w_2 = w'_2 - w'_1$ is in $W$ as a linear combination of vectors in $W$ and in $W'$ as a linear combination of vectors in $W'$. It follows $w \in W \cap W' = \{0_V\}$ that is $w = 0_V$. Consequently, $w_1 = w_2$ and $w'_1 = w'_2$. In other words, every vector $v \in \Sigma$ may be written uniquely as a sum of a vector in $W$ and a vector in $W'$, that is $\Sigma = W \oplus W'$ ■

**Example:** In $\mathbb{C}$ seen as a $\mathbb{R}$-vector space, the subspaces $\mathbb{R}$ and $\imath\mathbb{R} = \{\imath b \,/\, b \in \mathbb{R}\}$ are such that $\mathbb{R} \cap \imath\mathbb{R} = \{0\}$. In particular, $\mathbb{C} = \mathbb{R} \oplus \imath\mathbb{R}$, and every complex number $z \in \mathbb{C}$ may be written uniquely as $z = a + \imath b$, where $a \in \mathbb{R}$ is called the **real part** of $z$ and $b \in \mathbb{R}$ is called the **imaginary part** of $z$.

**Definition 3.14 (*Supplementary subspaces*)**

*Two subspaces $W$ and $W'$ of a $\mathbb{K}$-vector space $(V, +, .)$ are said* ***supplementary subspaces*** *in $V$ if $V = W \oplus W'$ or equivalently if*

$$\forall v \in V,\ \exists ! (w, w') \in W \times W' /\ v = w + w'$$

*In this case, we also say that $W'$ is a* ***supplementary subspace*** *of $W$ in $V$.*

**Example :** Consider the sets of all even or odd functions from $\mathbb{R}$ to itself:

$$\begin{aligned}\mathcal{E}(\mathbb{R}) &= \{f:\mathbb{R}\to\mathbb{R} \text{ even}\} = \{f:\mathbb{R}\to\mathbb{R} \ / \ \forall x\in\mathbb{R},\ f(-x)=f(x)\}\\ \mathcal{O}(\mathbb{R}) &= \{f:\mathbb{R}\to\mathbb{R} \text{ odd}\} = \{f:\mathbb{R}\to\mathbb{R} \ / \ \forall x\in\mathbb{R},\ f(-x)=-f(x)\}\end{aligned}$$

It follows from Proposition 3.6 that $\mathcal{E}(\mathbb{R})$ and $\mathcal{O}(\mathbb{R})$ are subspaces of $\mathcal{F}(\mathbb{R})$. Furthermore they are supplementary subspaces in $\mathcal{F}(\mathbb{R})$.

**Proof :** Let $f$ be a function from $\mathbb{R}$ to itself. Define the two following maps

$$\begin{array}{rcl} f_{\mathcal{E}}:\mathbb{R} & \longrightarrow & \mathbb{R}\\ x & \longmapsto & \frac{1}{2}\Big(f(x)+f(-x)\Big)\end{array} \quad \text{and} \quad \begin{array}{rcl} f_{\mathcal{O}}:\mathbb{R} & \longrightarrow & \mathbb{R}\\ x & \longmapsto & \frac{1}{2}\Big(f(x)-f(-x)\Big)\end{array}$$

For every $x\in\mathbb{K}$, we have:

$$\begin{cases} f_{\mathcal{E}}(-x) = \frac{1}{2}\Big(f(-x)+f(x)\Big) = \frac{1}{2}\Big(f(x)+f(-x)\Big) = f_{\mathcal{E}}(x)\\ f_{\mathcal{O}}(-x) = \frac{1}{2}\Big(f(-x)-f(x)\Big) = -\frac{1}{2}\Big(f(x)-f(-x)\Big) = -f_{\mathcal{O}}(x)\\ f_{\mathcal{E}}(x)+f_{\mathcal{O}}(x) = \frac{1}{2}\Big(f(x)+f(-x)+f(x)-f(-x)\Big) = \frac{1}{2}\Big(2f(x)\Big) = f(x)\end{cases}$$

Then $f=f_{\mathcal{E}}+f_{\mathcal{O}}$ with $f_{\mathcal{E}}$ even and $f_{\mathcal{O}}$ odd. Consequently $\mathcal{F}(\mathbb{R})=\mathcal{E}(\mathbb{R})+\mathcal{O}(\mathbb{R})$.
Now let $f$ be a function from $\mathbb{R}$ to itself which is odd and even. For every $x\in\mathbb{R}$, we have:

$$f(x)=f(-x)=-f(x)$$

Then $2f(x)=0$ that is $f(x)=0$. Consequently $\mathcal{E}(\mathbb{R})\cap\mathcal{O}(\mathbb{R})=\{0_{\mathcal{F}(\mathbb{R})}\}$. The conclusion follows from Proposition 3.13 ∎

**Remark :** A supplementary subspace of a given subspace is not necessarily unique (if there exists). For instance consider the real line $\mathbb{R}.v$ in the real plane $\mathbb{R}^2$ with $v=(1,0)\in\mathbb{R}^2$. Then $v'=(0,1)$ and $v''=(1,1)$ give two distinct supplementary subspaces $\mathbb{R}.v'$ and $\mathbb{R}.v''$ of $\mathbb{R}.v$. Actually, we have $\mathbb{R}^2=\mathbb{R}.v\oplus\mathbb{R}.w$ for every non zero vector $w\in\mathbb{R}^2$ which is non colinear with $v$ (that is $w\notin\mathbb{R}.v$).

# 3.3 Linear map

## 3.3.1 Definition and examples

**Definition 3.15 (*Linear map*)**

*Let $(V,+,.)$ and $(W,+,.)$ be two $\mathbb{K}$-vector spaces. A **linear map** from $(V,+,.)$ to $(W,+,.)$ is a function $f:V\to W$ such that*

$$\begin{cases} \forall (v,w)\in V^2, & f(v+w)=f(v)+f(w)\\ \forall\lambda\in\mathbb{K},\ \forall v\in V, & f(\lambda.v)=\lambda.f(v)\end{cases}$$

*In particular, the image of a linear combination of vectors in $V$ under a linear map $f:V\to W$ is a linear combination of vectors in $W$, that is*

$$\forall n\in\mathbb{N},\ \forall(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{K}^n,\ \forall(v_1,v_2,\ldots,v_n)\in V^n,$$
$$f\left(\sum_{k=1}^{n}\lambda_k.v_k\right)=\sum_{k=1}^{n}\lambda_k.f(v_k)$$

**Remark :** In particular, $n = 0$ gives $f(0_V) = 0_W$.

**Proof :** We have:

$$f(0_V) + f(0_V) = f(0_V + 0_V) = f(0_V)$$

Now, a simplification on each side by $f(0_V)$ gives $f(0_V) = 0_W$ as needed. ■

**Proposition 3.16**

*Let $(V, +, .)$ and $(W, +, .)$ be two $\mathbb{K}$-vector spaces. A function $f : V \to W$ is a linear map if and only if it satisfies the following condition*

$$\forall \lambda \in \mathbb{K},\ \forall (v, w) \in V^2,\ \ f(v + \lambda.w) = f(v) + \lambda.f(w)$$

**Proof : Necessary.** If $f$ is a linear map then

$$f(v + \lambda.w) = f(v) + f(\lambda.w) = f(v) + \lambda.f(w)$$

**Suficient.** $\lambda = 1_{\mathbb{K}}$ gives $f(v+w) = f(v)+f(w)$ for every vectors $v$ and $w$ in $V$. And $v = 0_V$ gives $f(\lambda.w) = f(0_V) + \lambda.f(w) = 0_W + \lambda.f(w) = \lambda.f(w)$ for every $\lambda \in \mathbb{K}$ and $w \in V$. ■

**Examples :**

**a)** The constant function $f : v \mapsto f(v) = 0_V$ equal to the zero vector is a linear map from $(V, +, .)$ to itself (even onto the trivial subspace $\{0_V\}$).

**b)** The idendity function $f = \mathrm{Id}_V = (v \mapsto v)$ is a linear map from $(V, +, .)$ onto itself.

**c)** The homothety $f_\mu : v \mapsto \mu.v$ of ratio $\mu \in \mathbb{K}$ is a linear map from $(V, +, .)$ onto itself.

**Proof :** For every $\lambda \in \mathbb{K}$ and every $(v, w) \in V^2$ we have:

$$\mu.(v + \lambda.w) = \mu.v + \mu.(\lambda.w) = \mu.v + (\mu\lambda).w = \mu.v + (\lambda\mu).w = \mu.v + \lambda.(\mu.w)$$

Consequently $f_\mu(v + \lambda.w) = f_\mu(v) + \lambda.f_\mu(w)$ and the conclusion follows from Proposition 3.16. ■

**d)** Consider the following map

$$f :\ v = (x, y, z) \longmapsto f(v) = (2x - y, 3z)$$

Then $f$ is a linear map from the real space $\mathbb{R}^3$ to the real plane $\mathbb{R}^2$.

**e) Counterexample.** The function $f : x \mapsto x^2$ from the vector space $\mathbb{R}$ to itself is not a linear map. Indeed notice that $f(1) = 1$ but $f(1 + 1) = f(2) = 4$ is not equal to $1 + 1 = 2$.

**f) Counterexample.** The function $f : x \mapsto x + 1$ from the vector space $\mathbb{R}$ to itself is not a linear map. Actually $f(x) + f(0) = f(x) + 1$ is not equal to $f(x) = f(x + 0)$.

**Further examples :**

**a) Polynomial space.** The differentiation map $P \mapsto P'$ is a linear map from $\mathbb{K}[X]$ to itself. Given $\alpha \in \mathbb{K}$, the map $P \mapsto P(\alpha)$ is a linear map from $\mathbb{K}[X]$ to $\mathbb{K}$ (seen as the coordinate space of dimension 1). But in general, a polynomial map $P : \mathbb{K} \to \mathbb{K}$ is not a linear map.

**b) Function space.** The differentiation map $f \mapsto f'$ is a linear map from the subspace of all differentiable functions to $\mathcal{F}(\mathbb{R})$. Given $(a, b) \in \mathbb{R}^2$, the integral $f \mapsto \int_a^b f(t)dt$ is a linear map from the subspace of all continuous functions to $\mathbb{R}$.

**c) Sequence space.** Consider the following maps from $\mathcal{F}(\mathbb{N},\mathbb{K})$ to itself

$$\begin{array}{rcl} S_L : u = (u_0, u_1, u_2, \dots) & \longmapsto & S_L(u) = (u_1, u_2, u_3, \dots) \\ S_R : u = (u_0, u_1, u_2, \dots) & \longmapsto & S_R(u) = (0_{\mathbb{K}}, u_0, u_1, \dots) \end{array}$$

Then $S_L$ and $S_R$ are linear maps called respectively the **left hand side shift** and the **right hand side shift**.

### 3.3.2 Linear map and subspaces

**Proposition 3.17**

*Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces. Then the following properties hold*

1. *If $V'$ is a subspace of $V$ then the image of $V'$ under $f$ defined as follows*

$$f(V') = \{f(v') \in W \ / \ v' \in V'\}$$

*is a subspace of $W$.*

2. *if $W'$ is a subspace of $W$ then the inverse image of $W'$ under $f$ defined as follows*

$$f^{-1}(W') = \{v \in V \ / \ f(v) \in W'\}$$

*is a subspace of $V$.*

**Proof:** 1. Let $\lambda$ be a scalar in $\mathbb{K}$ and $w_1, w_2$ be two vectors in $f(V')$. Then there exist two vectors $v_1', v_2'$ in $V'$ such that $w_1 = f(v_1')$ and $w_2 = f(v_2')$. Consequently we get

$$w_1 + \lambda.w_2 = f(v_1') + \lambda.f(v_2') = f(v_1' + \lambda.v_2') \in f(V')$$

since $v_1' + \lambda.v_2' \in V'$ as a linear combination of vector in $V'$. The result follows from Proposition 3.6.

2. Let $\lambda$ be a scalar in $\mathbb{K}$ and $v_1, v_2$ be two vectors in $f^{-1}(W')$. In particular $f(v_1), f(v_2)$ are two vectors in $W'$. Consequently, we get

$$f(v_1 + \lambda.v_2) = f(v_1) + \lambda.f(v_2) \in W'$$

as a linear combination of vectors in $W'$. Then $v_1 + \lambda.v_2 \in f^{-1}(W')$ and the result follows from Proposition 3.6. ∎

**Definition 3.18 (*Image and Kernel*)**

*Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces.*

- *The **image** of $f$ is defined to be the following subspace of $W$*

$$\mathrm{Im}(f) = f(V) = \{f(v) \in W \ / \ v \in V\}$$

- *The **kernel** of $f$ is defined to be the following subspace of $V$*

$$\mathrm{Ker}(f) = f^{-1}(\{0_W\}) = \{v \in V \ / \ f(v) = 0_W\}$$

**Proposition 3.19**

*Let $f : V \to W$ be a linear map between two $\mathbb{K}$-vector spaces. Then the following equivalence hold*

1. *$f$ is a surjective map if and only if $\mathrm{Im}(f) = W$*
2. *$f$ is an injective map if and only if $\mathrm{Ker}(f) = \{0_V\}$*

**Proof:** 1. **Necessary.** If $f$ is surjective then every vector $w \in W$ may be written as $w = f(v)$ with $v \in V$, that is $W = \mathrm{Im}(f)$.

**Sufficient.** if $\mathrm{Im}(f) = W$ then for every vector $w \in W$ there exists $v \in V$ such that $f(v) = w$, that is $f$ is surjective.

2. **Necessary.** If $f$ is injective then $f(v) = 0_W = f(0_V)$ implies $v = 0_V$, that is $\mathrm{Ker}(f) = \{0_V\}$.

**Sufficient.** Assume $\mathrm{Ker}(f) = \{0_V\}$ and let $v_1, v_2$ be two vectors in $V$ such that $f(v_1) = f(v_2)$.

$$f(v_1 - v_2) = f\left(v_1 + (-1_{\mathbb{K}}).v_2\right) = f(v_1) + (-1_{\mathbb{K}}).f(v_2) = f(v_1) - f(v_2) = 0_W$$

Then $v_1 - v_2 \in \mathrm{Ker}(f) = \{0_V\}$, that is $v_1 - v_2 = 0_V$ or equivalently $v_1 = v_2$. It follows that $f$ is injective. ■

**Example:** The left hand side shift map $S_L$ is surjective whereas the right hand side shift map $S_R$ is not. But $S_R$ is injective whereas $S_L$ is not (indeed $\mathrm{Ker}(S_L) = \{u = (u_0, 0_{\mathbb{K}}, 0_{\mathbb{K}}, 0_{\mathbb{K}}, \dots) \ / \ u_0 \in \mathbb{K}\}$).

**Definition 3.20 (*Projection*)**

*Let $W$ and $W'$ be two supplementary subspaces of a $\mathbb{K}$-vector space $V$, that is $V = W \oplus W'$. Since every vector $v \in V$ may be written uniquely as $v = w + w'$ with $w \in W$ and $w' \in W'$, the following map is well defined.*

$$\begin{array}{cccc} p: & V & \longrightarrow & V \\ & v = w + w' & \longmapsto & p(v) = w \end{array}$$

*The linear map $p$ is called the* ***projection along W' onto W****.*

**Example:** In $\mathbb{C}$ seen as a $\mathbb{R}$-vector space, we have $\mathbb{C} = \mathbb{R} \oplus \imath\mathbb{R}$. So, we may define

$$\begin{array}{cccc} \Re: & \mathbb{C} & \longrightarrow & \mathbb{R} \\ & z = a + \imath b & \longmapsto & \Re(z) = a \end{array} \quad \text{and} \quad \begin{array}{cccc} \Im: & \mathbb{C} & \longrightarrow & \mathbb{R} \\ & z = a + \imath b & \longmapsto & \Im(z) = b \end{array}$$

The real part function $\Re$ is the projection along $\imath\mathbb{R}$ onto $\mathbb{R}$. And the projection along $\mathbb{R}$ onto $\imath\mathbb{R}$ is given by $\imath\Im : z = a + \imath b \mapsto \imath b$ where $\Im$ is the imaginary part function.

**Proposition 3.21**

*Let $W$ and $W'$ be two supplementary subspaces of a $\mathbb{K}$-vector space $V$, that is $V = W \oplus W'$. If $p$ is the projection along W' onto W then*

$$\mathrm{Im}(p) = W \quad \textit{and} \quad \mathrm{Ker}(p) = W'$$

*In particular, if $p$ is a projection then $V = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.*

**Proof:** It follows from the definition of a projection that $\mathrm{Im}(p) \subset W$ and $W' \subset \mathrm{Ker}(p)$. If $w$ is a vector in $W$ then $p(w) = p(w + 0_V) = w$, that is $w = p(w) \in \mathrm{Im}(p)$. Consequently $W \subset \mathrm{Im}(p)$. Now if $v$ is a vector in $\mathrm{Ker}(p)$, let $w \in W$ and $w' \in W'$ be such that $v = w + w'$, then $0_V = p(v) = p(w + w') = w$. It follows that $v = 0_V + w' = w' \in W'$ and consequently $\mathrm{Ker}(p) \subset W'$. ■

### 3.3.3 Isomorphic vector spaces

**Definition 3.22** (***Isomorphic vector spaces***)

*Two $\mathbb{K}$-vector spaces $V$ and $W$ are said **isomorphic** if there exists a bijective linear map $f$ from $V$ to $W$. In this case, we write $\boldsymbol{V \approx W}$ and $f$ is called a **vector space isomorphism**.*

**Example :** Consider the following function

$$\begin{array}{cccc} \varphi : & \mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ & z = a + \imath b & \longmapsto & \varphi(z) = (a, b) \end{array}$$

Then $\varphi$ is a linear map since $\varphi : z \mapsto (\Re(z), \Im(z))$ where $\Re$ and $\Im$ are linear maps. Moreover $\varphi$ is bijective. Consequently, the $\mathbb{R}$-vector spaces $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic.

**Further example :** The $\mathbb{K}$-vector space $\mathbb{K}[X]$ of all polynomials with coefficients in $\mathbb{K}$ is isomorphic to the subspace of $\mathcal{F}(\mathbb{N}, V)$ of all infinite sequences of elements in $\mathbb{K}$ which are eventually equal to zero.

### 3.3.4 Sets of linear maps

**Definition 3.23** (***Sets of linear maps***)

*Let $V$ and $W$ be two $\mathbb{K}$-vector spaces.*

- *We denote by $\boldsymbol{L(V, W)}$ (or $\mathbf{Hom}\boldsymbol{(V, W)}$) the set of all linear maps from $V$ to $W$, that is the set of all **vector space homomorphisms** from $(V, +, .)$ to $(W, +, .)$.*
- *We denote by $\boldsymbol{L(V)}$ (or $\mathbf{End}\boldsymbol{(V)}$) the set of all linear maps from $V$ to itself, that is the set of all **vector space endomorphisms** of $(V, +, .)$.*
- *We denote by $\boldsymbol{GL(V)}$ (or $\mathbf{Aut}\boldsymbol{(V)}$) the set of all bijective linear maps from $V$ to itself, that is the set of all **vector space automorphisms** of $(V, +, .)$.*

**Remarks :**

- We also denote by $\mathbf{Isom}\boldsymbol{(V, W)}$ the set of all bijective linear maps from $V$ to $W$, that is the set of all vector space isomorphisms from $(V, +, .)$ to $(W, +, .)$. In particular, $V$ and $W$ are isomorphic if and only if $\mathrm{Isom}(V, W) \neq \emptyset$.
- If $f \in GL(V)$ then its bijective inverse $f^{-1} : V \to V$ is also a linear map, that is $f^{-1} \in GL(V)$.

  **Proof :** Let $\lambda$ be a scalar in $\mathbb{K}$ and $v, w$ be two vectors in $V$. Since $f$ is a surjective function, there exist two vectors $v', w'$ in $V$ such that $f(v') = v$ and $f(w') = w$. Then

  $$f^{-1}(v + \lambda.w) = f^{-1}(f(v') + \lambda.f(w')) = f^{-1}(f(v' + \lambda.w')) = v' + \lambda.w' = f^{-1}(v) + \lambda.f^{-1}(w)$$

  The conclusion follows from Proposition 3.16. ■

**Proposition 3.24**

*Let $V$ and $W$ be two $\mathbb{K}$-vector spaces. For every scalar $\lambda \in \mathbb{K}$ and every linear maps $f, g$ from $V$ to $W$, the maps $f+g$ and $\lambda.f$ defined as follows*

$$\begin{array}{rcl} f+g : V & \longrightarrow & W \\ v & \longmapsto & (f+g)(v) = f(v) + g(v) \end{array} \quad \text{and} \quad \begin{array}{rcl} \lambda.f : V & \longrightarrow & W \\ v & \longmapsto & (\lambda.f)(v) = \lambda.f(v) \end{array}$$

*are linear maps from $V$ to $W$. That provides an addition and a scalar multiplication on $L(V,W)$, and the following properties hold*

1. *$(L(V,W), +, .)$ is a $\mathbb{K}$-vector space.*
2. *$(L(V), +, .)$ is a $\mathbb{K}$-vector space.*

**Proof:** The second point follows from the first one with $V = W$. Now, let $f$ and $g$ be two linear maps from $V$ to $W$ and $\lambda$ be a scalar in $\mathbb{K}$. Then for every vector $v, w$ in $V$ and every scalar $\mu$ in $\mathbb{K}$, we have:

$$\begin{array}{rcl} (f+\lambda.g)(v+\mu.w) & = & f(v+\mu.w) + \lambda.g(v+\mu.w) \\ & = & f(v) + \mu.f(w) + \lambda.(g(v) + \mu.g(w)) \\ & = & f(v) + \mu.f(w) + \lambda.g(v) + \lambda.(\mu.g(w)) \\ & = & f(v) + \lambda.g(v) + \mu.f(w) + (\lambda\mu).g(w) \\ & = & (f+\lambda.g)(v) + \mu.f(w) + (\mu\lambda).g(w) \\ & = & (f+\lambda.g)(v) + \mu.f(w) + \mu.(\lambda.g(w)) \\ & = & (f+\lambda.g)(v) + \mu.(f(w) + \lambda.g(w)) \\ & = & (f+\lambda.g)(v) + \mu.(f+\lambda.g)(w) \end{array}$$

It follows from Proposition 3.16 that $f + \lambda.g$ is a linear map and from Proposition 3.6 that $L(V,W)$ is a subspace of the $\mathbb{K}$-vector space $\mathcal{F}(V,W)$ of all functions from $V$ to $W$. In particular, $L(V,W)$ is a $\mathbb{K}$-vector space. ■

**Remark:** But a linear combination of bijective linear maps is not always bijective.

**Proposition 3.25**

*Let $U$, $V$ and $W$ be three $\mathbb{K}$-vector spaces. For every linear map $f$ from $V$ to $W$ and every linear map $g$ from $U$ to $V$, the composition map $f \circ g$ defined as follows*

$$\begin{array}{rcl} f \circ g : U & \longrightarrow & W \\ u & \longmapsto & (f \circ g)(u) = f(g(u)) \end{array}$$

*is a linear map from $U$ to $W$. In case $U = V = W$, that provides a binary operation on $L(V)$, and the following properties hold*

1. *$(L(V), +, \circ)$ is an unital ring.*
2. *$(GL(V), \circ)$ is a group.*

*The identity elements are respectively the constant linear map equal to $0_V$ for addition and the identity function $\mathrm{Id}_V$ for composition.*

**Proof:** The second point follows from the first one and from the claim that the bijective inverse of a linear map is also a linear map. For the first point, we have:

1. $(L(V), +)$ is an abelian group.
2. $\circ$ is associative.
3. $\circ$ is distributive over $+$. Indeed for every linear maps $f, g, h$ from $V$ to $V$, we have:

$$\forall v \in V, \begin{cases} f((g+h)(v)) = f(g(v)+h(v)) = f(g(v)) + f(h(v)) \\ (g+h)(f(v)) = g(f(v)) + h(f(v)) \end{cases}$$

   that is
$$\begin{cases} f \circ (g+h) = f \circ g + f \circ h \\ (g+h) \circ f = g \circ f + h \circ f \end{cases}$$

4. The identity map $\mathrm{Id}_V : v \mapsto v$ is the identity element for $\circ$.

The result follows. ■

**Remark:** But the composition of linear maps is not commutative.