University of Cergy-Pontoise **2010-2011**
L1-MI 2nd semester

Test n° 2 - Answers and Solutions

Question about course 1. The subspaces E_1 and E_2 are supplementary in V if they satisfy the following conditions

i) $V = E_1 + E_2$, that is $\forall v \in V,\ \exists (v_1, v_2) \in E_1 \times E_2 \ / \ v = v_1 + v_2$

ii) $E_1 \cap E_2 = \{0_V\}$

In this case one can write $V = E_1 \oplus E_2$, that is $\forall v \in V,\ \exists! (v_1, v_2) \in E_1 \times E_2 \ / \ v = v_1 + v_2$

Problem 2. a)

$$\begin{array}{rcllllllll}
P(X) & = & X^4 & -4X^3 & +9X^2 & -20X & +20 & & \\
P(2) & = & 16 & -32 & +36 & -40 & +20 & = & 0 \\
P'(X) & = & 4X^3 & -12X^2 & +18X & -20 & & & \\
P'(2) & = & 32 & -48 & +36 & -20 & & = & 0 \\
P^{(2)}(X) & = & 12X^2 & -24X & +18 & & & & \\
P^{(2)}(2) & = & 48 & -48 & +18 & & & = & 18 \neq 0
\end{array}$$

Then 2 is a root of P of multiplicity 2.

b) (1) For every $t \in \mathbb{R}$, we have

$$\begin{array}{rcl}
P(it) & = & i^4t^4 \quad -4i^3t^3 \quad +9i^2t^2 \quad -20it \quad +20 \\
& = & t^4 \quad +4it^3 \quad -9t^2 \quad -20it \quad +20 \\
& = & (t^4 - 9t^2 + 20) + (4it^3 - 20it) \\
& = & (t^4 - 9t^2 + 20) + 4it(t^2 - 5)
\end{array}$$

(2) By factorizing real and imaginary parts of $P(it)$, we get

$$\Re(P(it)) = t^4 - 9t^2 + 20 = (t^2 - 4)(t^2 - 5) = (t - \sqrt{4})(t + \sqrt{4})(t - \sqrt{5})(t + \sqrt{5})$$

$$\Im(P(it)) = 4t(t^2 - 5) = 4t(t - \sqrt{5})(t + \sqrt{5})$$

Consequently

$$\begin{array}{rcl}
P(it) = 0 & \Leftrightarrow & \left\{ \begin{array}{l} \Re(P(it)) = 0 \\ \Im(P(it)) = 0 \end{array} \right. \\
& \Leftrightarrow & \left\{ \begin{array}{l} t = \sqrt{4} \quad \text{or} \quad t = -\sqrt{4} \quad \text{or} \quad t = \sqrt{5} \quad \text{or} \quad t = -\sqrt{5} \\ t = 0 \quad \text{or} \quad t = \sqrt{5} \quad \text{or} \quad t = -\sqrt{5} \end{array} \right. \\
& \Leftrightarrow & t = \sqrt{5} \quad \text{or} \quad t = -\sqrt{5}
\end{array}$$

Equivalently, it_0 and $-it_0$ with $t_0 = \sqrt{5} > 0$ are roots of P.

c) We got three roots of P, one of which is of multiplicity 2. Consequently P may be written as

$$P(X) = (X-2)^2(X-it_0)(X+it_0)Q(X)$$

Moreover P and $(X-2)^2(X-it_0)(X+it_0)$ are of degree 4, then $Q(X)$ is of degree 1, that is a constant polynomial equal to the leading coefficient of P which is 1. Finally we have

$$P(X) = (X-2)^2(X-it_0)(X+it_0) = (X-2)^2(X^2+t_0^2)$$

Problem 3. a) (1) We have

$$\begin{cases} x>0 \\ y>0 \end{cases} \Rightarrow \begin{cases} \ln(x)\in\mathbb{R} \\ \ln(y)\in\mathbb{R} \end{cases} \Rightarrow \ln(x)\ln(y)\in\mathbb{R} \Rightarrow x\star y = e^{\ln(x)\ln(y)} > 0$$

That is $\star$ is a binary operation on $]0,+\infty[$

(2) Since the multiplication of real numbers is commutative, we get for every positive real numbers x and y :

$$x\star y = e^{\ln(x)\ln(y)} = e^{\ln(y)\ln(x)} = y\star x$$

Then $\star$ is commutative.

(3) For every positive real numbers x, y and z, we have

$$\begin{cases} (x\star y)\star z &= e^{\ln(x)\ln(y)}\star z &= e^{\ln\left(e^{\ln(x)\ln(y)}\right)\ln(z)} &= e^{\ln(x)\ln(y)\ln(z)} \\ x\star(y\star z) &= x\star e^{\ln(y)\ln(z)} &= e^{\ln(x)\ln\left(e^{\ln(y)\ln(z)}\right)} &= e^{\ln(x)\ln(y)\ln(z)} \end{cases}$$

Consequently $(x\star y)\star z = x\star(y\star z)$ that is $\star$ is associative.

b) For every $x>0$, we have

$$\begin{cases} e\star x = e^{\ln(e)\ln(x)} = e^{\ln(x)} = x & \text{since } \ln(e)=1 \\ x\star e = e\star x = x & \text{since } \star \text{ is commutative} \end{cases}$$

Then e is the identity element for $\star$.

c) (1) For every $x\in E =]0,+\infty[-\{1\}$, we have

$$x\star y = y\star x = e \Rightarrow e^{\ln(x)\ln(y)} = e^1 \Rightarrow \ln(x)\ln(y) = 1 \Rightarrow \ln(y) = \frac{1}{\ln(x)} \Rightarrow y = e^{\frac{1}{\ln(x)}}$$

and y is well defined because $x\neq 1 \Rightarrow \ln(x)\neq 0$. Moreover $y = e^{\frac{1}{\ln(x)}} > 0$ and $y\neq 1$ because $\frac{1}{\ln(x)}\neq 0$.

Consequently $x\in E$ has an inverse element $x^{-1}\in E$ which is $x^{-1} = e^{\frac{1}{\ln(x)}}$.

(2) Finally $(E,\star)$ is an abelian group.