University of Cergy-Pontoise **2010-2011**
L1-MI 2^{nd} semester

Test n° 1

(1 hour)

Question about course 1. Recall the axioms that a ring $(R, +, \times)$ must satisfy.

Exercise 2. Prove by induction the following claim for every integer $n \in \mathbb{N}$

$$6^{2n+1} + 2^{3n+1} \equiv 1 \quad [7]$$

Problem 3. Consider the following set

$$A = \left\{ a + b\sqrt{2} \;/\; (a, b) \in \mathbb{Z}^2 \right\}$$

1. Assuming $\sqrt{2} \notin \mathbb{Q}$ (that is there exist no integers p and q such that $\sqrt{2} = \frac{p}{q}$), show that

$$\forall a + b\sqrt{2} \in A, \quad a + b\sqrt{2} = 0 \Longleftrightarrow a = b = 0$$

2. Prove that $(A, +, \times)$ is a commutative unital ring but not a field.
3. Consider the following maps

$$\begin{array}{rrcl} \varphi : & A & \longrightarrow & A \\ & a + b\sqrt{2} & \longmapsto & a - b\sqrt{2} \end{array} \quad \text{and} \quad \begin{array}{rrcl} N : & A & \longrightarrow & A \\ & x & \longmapsto & x \times \varphi(x) \end{array}$$

   (a) Prove that

$$\begin{cases} \forall x \in A, & N(x) \in \mathbb{Z} \\ \forall (x, y) \in A^2, & N(xy) = N(x)N(y) \end{cases}$$

   (b) Deduce that

$$\forall x \in A, \quad x \text{ has a multiplicative inverse in } A \Longleftrightarrow N(x) = 1 \text{ or } -1$$

Problem 4. Consider the following map for some parameter $\lambda \in \mathbb{R}^*$

$$\begin{array}{rrcl} f_\lambda : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x, y) & \longmapsto & \left(\lambda x, \dfrac{y}{\lambda} \right) \end{array}$$

1. Prove that f_λ is a group endomorphism from $(\mathbb{R}^2, +)$ to itself.
2. Consider the following set

$$\mathcal{F} = \{ f_\lambda \;/\; \lambda \in \mathbb{R}^* \}$$

   (a) Determine $f_\lambda \circ f_\mu$ for any λ and μ in $\mathbb{R}^*$.
   (b) Deduce that $\circ$ is a binary operation on $\mathcal{F}$.
   (c) Prove that $(\mathcal{F}, \circ)$ is a group.
3. Consider the following map

$$\begin{array}{rrcl} h : & \mathbb{R}^* & \longrightarrow & \mathcal{F} \\ & \lambda & \longmapsto & f_\lambda \end{array}$$

   (a) Prove that h is a group isomorphism from $(\mathbb{R}^*, \times)$ to $(\mathcal{F}, \circ)$.
   (b) Deduce that $(\mathcal{F}, \circ)$ is an abelian group.