Continuous assessment n°1

Exercice 1 For all vector $(x, y, z) \in \mathbb{R}^3$ we define :

$$N(x, y, z) = max\left(|x| + |y|, |z|\right)$$

1. Prove that N is a norm on $\mathbb{R}^3$.
2. Prove that N is equivalent to the norm $\|.\|_1 : (x, y, z) \mapsto \|(x, y, z)\|_1 = |x| + |y| + |z|$.

Exercice 2 We consider the set :

$$\Delta = \left\{(x, y) \in \mathbb{R}^2 / x = y\right\}$$

1. Let $(E, \|.\|)$ be a normed vector space. Give the definition of an open set and a closed set in E.
2. Is Δ open in $\mathbb{R}^2$? Is it closed in $\mathbb{R}^2$? Prove your answers.
3. Give an example of a subset of $\mathbb{R}$ which is neither open nor closed and explain why.

Exercice 3 Let f be the following function :

$$f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \begin{cases} \frac{(x-1)^3 y^2}{(x^2+y^2-2x+1)^2} & \text{si } (x, y) \neq (1, 0) \\ 0 & \text{si } (x, y) = (1, 0) \end{cases}$$

Study the continuity of f on $\mathbb{R}^2$.

Exercice 4 Let g be the following funtion :

$$g : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^2} & \text{si } (x, y) \neq (0, 0) \\ 0 & \text{si } (x, y) = (0, 0) \end{cases}$$

Is g continuous at $(0, 0)$? Explain your answer.