Construction de fractions rationnelles à dynamique prescrite

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Université de Cergy-Pontoise

Soutenance de thèse - 12 mai 2010

Construction of rational maps with prescribed dynamics

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Thesis defense - May 12, 2010

Field: Study of holomorphic dynamical systems

Motivation: Find some examples of rational maps
with particular complicated dynamics

Questions:
1- How to construct rational maps
from dynamical informations ?
2- Which kind of rational maps
is it possible to construct ?

Main tools: Quasiconformal surgery and Thurston theory

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$.

For every $z_0 \in \widehat{\mathbb{C}}$, consider its forward orbit $\{z_n = f^{\circ n}(z_0) \,/\, n \geqslant 1\}$.

$$z_0 \overset{f}{\mapsto} z_1 \overset{f}{\mapsto} z_2 \overset{f}{\mapsto} z_3 \overset{f}{\mapsto} \dots$$

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Definition (Fatou and Julia sets)

- the Fatou set is

$$\mathcal{F}(f) = \{z_0 \in \widehat{\mathbb{C}} \,/\, (f^{\circ n})_{n \geqslant 1} \text{ is a normal family at } z_0\}$$

- the Julia set is

$$\mathcal{J}(f) = \widehat{\mathbb{C}} - \mathcal{F}(f)$$

Theorem

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$.
$\mathcal{J}(f)$ is a nonempty fully invariant closed and perfect set.
Furthermore either

- *$\mathcal{J}(f)$ is connected,*
- *or else $\mathcal{J}(f)$ has uncountably many connected components.*

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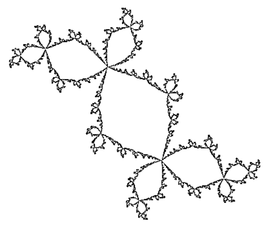
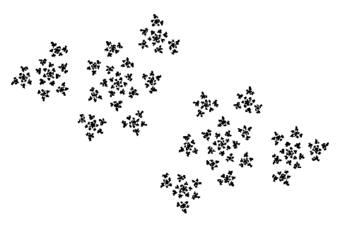
Example





Theorem

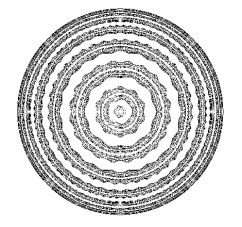
Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$.
If there exists an attracting fixed point z_∞ of f such that every critical point of f lies in the immediate attracting bassin of z_∞ then

$$\exists \text{ a homeomorphism } \phi \, / \quad \begin{array}{ccc} \mathcal{J}(f) & \xrightarrow{\;f\;} & \mathcal{J}(f) \\ \phi \downarrow & & \downarrow \phi \\ \Sigma_d & \xrightarrow[\sigma]{} & \Sigma_d \end{array}$$

where

- $\Sigma_d = \{1, 2, \ldots, d\}^{\mathbb{N}}$ *is a Cantor set*
- $\varepsilon = (\varepsilon_0 \varepsilon_1 \varepsilon_2 \ldots) \mapsto \sigma(\varepsilon) = (\varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots)$ *is the shift map*

Example (McMullen)



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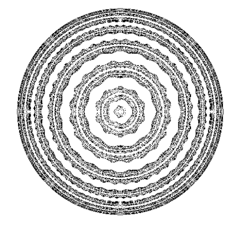


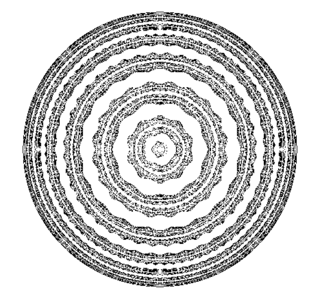
Theorem

f_{CoC} *acts on* $\mathcal{J}_{CoC} = \{J \text{ Julia component of } \mathcal{J}(f_{CoC})\} \approx \bigcup_{\alpha \in \Sigma_2} C_\alpha$.

$\exists$ *a homeomorphism* ϕ /

$$\begin{array}{ccc} \mathcal{J}_{CoC} & \xrightarrow{f_{CoC}} & \mathcal{J}_{CoC} \\ \downarrow \phi & & \downarrow \phi \\ \Sigma_2 & \xrightarrow[\sigma]{} & \Sigma_2 \end{array}$$

Example (McMullen)



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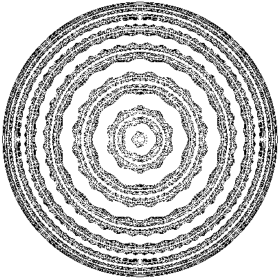

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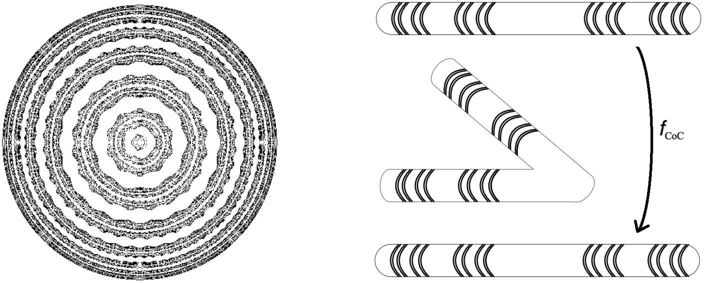

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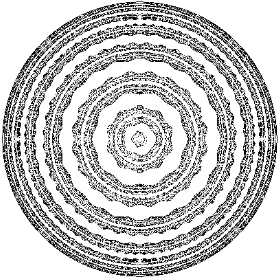
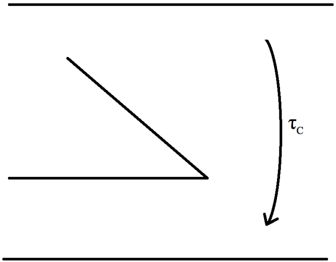




Theorem

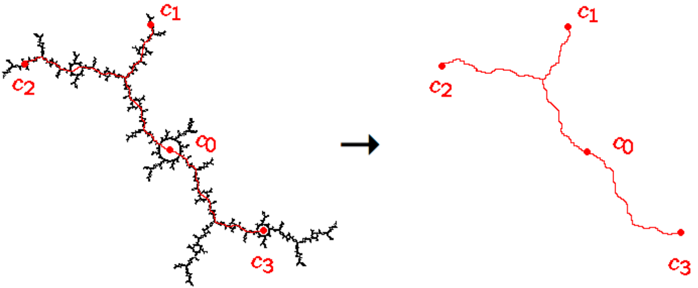
$\exists$ *a homeomorphism* φ /

$$\begin{array}{ccc} \mathcal{J}_{CoC} & \xrightarrow{f_{CoC}} & \mathcal{J}_{CoC} \\ \varphi \downarrow & & \downarrow \varphi \\ \mathcal{J}_C & \xrightarrow[\tau_C]{} & \mathcal{J}_C \end{array}$$

where

- $\tau_C : [0,1] \to [0,1],\ x \mapsto \begin{cases} 3x & \textit{if } x \in [0, \frac{1}{2}] \\ 3(1-x) & \textit{if } x \in [\frac{1}{2}, 1] \end{cases}$
- *and* $\mathcal{J}_C = \left\{ x \in [0,1] \,/\, \forall n \geqslant 0,\ \tau_C^{\circ n}(x) \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \right\}$

Consider $P_c : z \mapsto z^2 + c$ where $c \approx -0.157\ldots + 1.032\ldots i$



Let $\mathcal{J}_{\mathcal{H}}$ be the intersection between $\mathcal{J}(P_c)$ and the Hubbard tree $\mathcal{H}$

Theorem (Persian carpet)

There exists a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *such that*

$\exists$ *a homeomorphism* φ /

$$\begin{array}{ccc} \mathcal{J}_{\mathcal{H}}(f) & \xrightarrow{f} & \mathcal{J}_{\mathcal{H}}(f) \\ \varphi \downarrow & & \downarrow \varphi \\ \mathcal{J}_{\mathcal{H}} & \xrightarrow[P_c]{} & \mathcal{J}_{\mathcal{H}} \end{array}$$

where $\mathcal{J}_{\mathcal{H}}(f)$ *is a subset of Julia components of* f.
Moreover,

- *there exists only one fixed Julia component* J_α
- $\forall J \in \mathcal{J}_{\mathcal{H}}(f) - \bigcup_{n \geqslant 0} (f^{\circ n})^{-1}(J_\alpha)$, *$J$ is a Jordan curve*

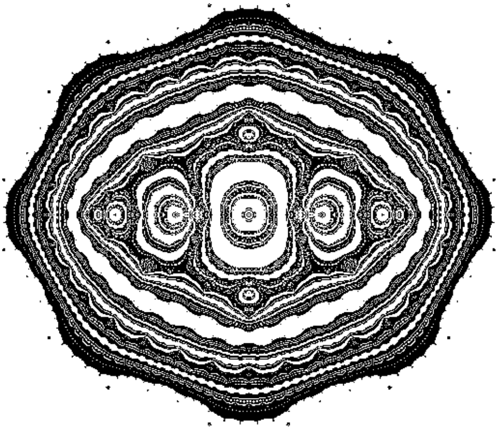
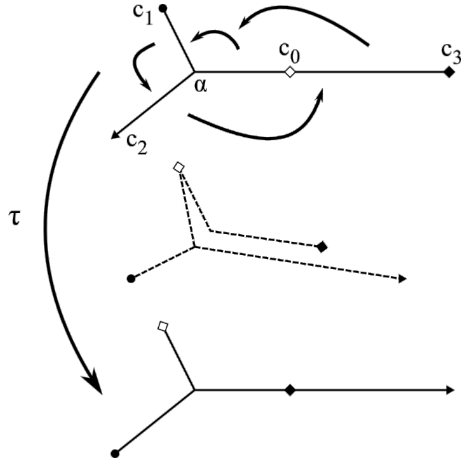
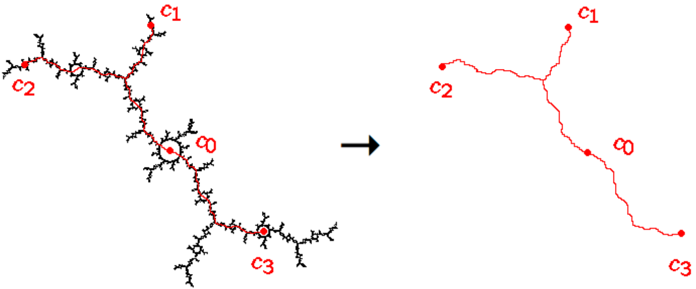


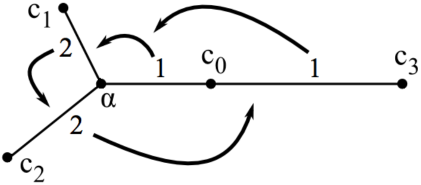
Figure: A Persian carpet

Consider the following abstract Hubbard tree $\mathcal{H} = (T, \tau)$.




c1
c2
c0
c3
c1
c2
c0
c3

We equip the Hubbard tree $\mathcal{H}$ with a weight function w.



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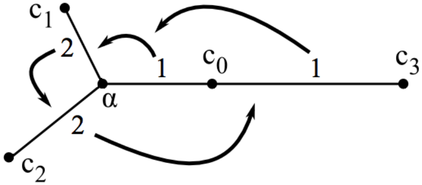
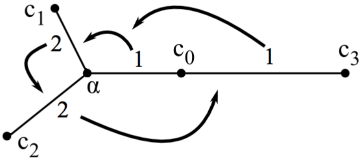
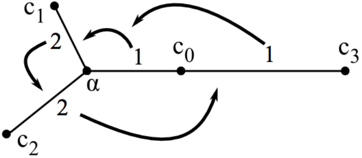


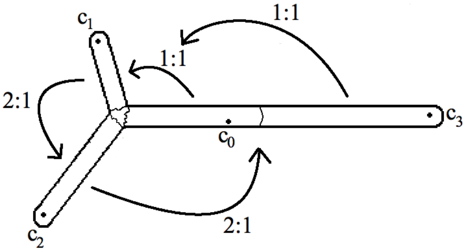
Fact (the weighted Hubbard tree $(\mathcal{H}, w)$ is unobstructed)

$$\begin{cases} \tau(e_{\alpha,c_0}) = e_{\alpha,c_1} \\ \tau(e_{\alpha,c_1}) = e_{\alpha,c_2} \\ \tau(e_{\alpha,c_2}) = e_{\alpha,c_0} \cup e_{c_0,c_3} \\ \tau(e_{c_0,c_3}) = e_{\alpha,c_1} \cup e_{\alpha,c_0} \end{cases} \quad \textit{gives} \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

with $\lambda(\mathcal{H}) := \lambda(M) \approx 0.918 < 1$


c1
2
1
c0
1
c3
α
2
c2


c_1
2
1
c_0
1
c_3
α
2
c_2


c_1
1:1
1:1
2:1
c_3
c_0
2:1
c_2

Question: How to construct a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ "encoded" by the unobstructed weighted Hubbard tree $(\mathcal{H}, w)$?

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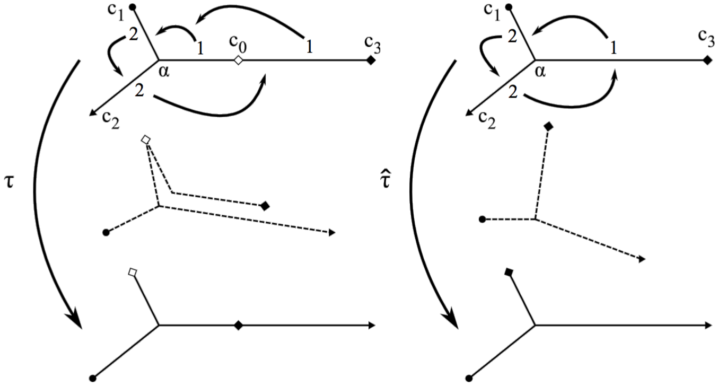
Answer: By quasiconformal surgery !


c_1
2
1
c_0
1
c_3
α
2
c_2
τ
c_1
2
1
c_3
α
2
c_2
$\hat{\tau}$

$$c_1 \xrightarrow{2} c_2 \xrightarrow{2} c_3 \xrightarrow{1} c_1$$

$$c_1 = 1 \xrightarrow{2} c_2 = \infty \xrightarrow{2} c_3 = 0 \quad (c_3 \xrightarrow{1} c_1)$$

$$\widehat{f} = (z \mapsto z^2) \circ \left(z \mapsto \frac{1}{1-z}\right) = \left(z \mapsto \frac{1}{(1-z)^2}\right)$$

$$c_1 = 1 \xrightarrow{2} c_2 = \infty \xrightarrow{2} c_3 = 0 \quad (c_3 \xrightarrow{1} c_1)$$

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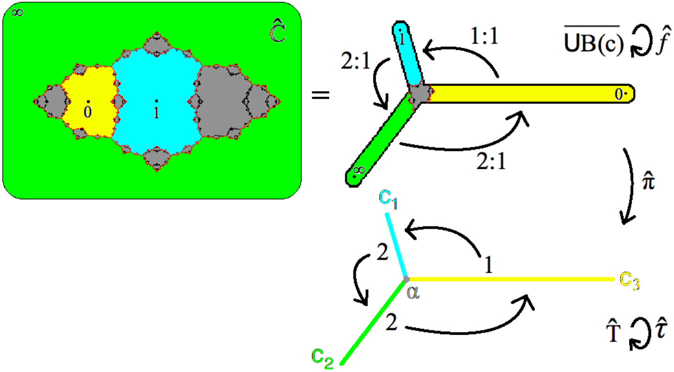


Step 1 - Cutting off

Lemma (equipotentials layout)

Given any positive constant $C > 0$, there exist five equipotentials β_0, β_1, β_2, γ_{-3} and γ_{+3} such that

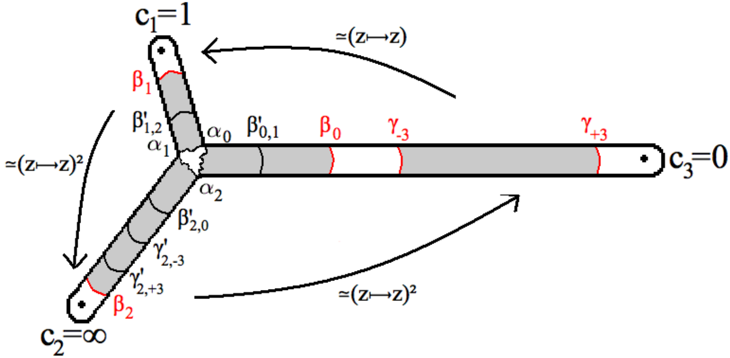
(i) $\beta_0 \subset B(0)$, $\beta_1 \subset B(1)$ *and* $\beta_2 \subset B(2)$

(ii) $\gamma_{-3}, \gamma_{+3} \subset B(0)$ *and* $|\phi_0(\beta_0)| > |\phi_0(\gamma_{-3})| > |\phi_0(\gamma_{+3})|$

(iii) *the following inequalities hold*

$$\begin{cases} \mathrm{mod}(\alpha_1, \beta_1) < \mathrm{mod}(\alpha_0, \beta_0) \\ \frac{1}{2}\mathrm{mod}(\alpha_2, \beta_2) < \mathrm{mod}(\alpha_1, \beta_1) \\ \frac{1}{2}\mathrm{mod}(\alpha_0, \beta_0) + \frac{1}{2}\mathrm{mod}(\gamma_{-3}, \gamma_{+3}) < \mathrm{mod}(\alpha_2, \beta_2) \\ \mathrm{mod}(\alpha_0, \beta_0) + \mathrm{mod}(\alpha_1, \beta_1) + C < \mathrm{mod}(\gamma_{-3}, \gamma_{+3}) \end{cases} \quad (1)$$

$$\frac{1}{2}\mathrm{mod}(\alpha_0, \gamma_{+3}) < \mathrm{mod}(\alpha_2, \beta_2) \quad (2)$$


c1=1
≃(z⟼z)
β1
β'1,2
α1
α0
β'0,1
β0
γ-3
γ+3
c3=0
≃(z⟼z)²
α2
β'2,0
γ'2,-3
γ'2,+3
β2
c2=∞
≃(z⟼z)²

Sketch of proof for equipotentials layout Lemma.

Compare

$$\left\{\begin{array}{rcl} \mathrm{mod}(\alpha_1,\beta_1) & < & \mathrm{mod}(\alpha_0,\beta_0) \\ \frac{1}{2}\,\mathrm{mod}(\alpha_2,\beta_2) & < & \mathrm{mod}(\alpha_1,\beta_1) \\ \frac{1}{2}\,\mathrm{mod}(\alpha_0,\beta_0)+\frac{1}{2}\,\mathrm{mod}(\gamma_{-3},\gamma_{+3}) & < & \mathrm{mod}(\alpha_2,\beta_2) \\ \mathrm{mod}(\alpha_0,\beta_0)+\mathrm{mod}(\alpha_1,\beta_1)+C & < & \mathrm{mod}(\gamma_{-3},\gamma_{+3}) \end{array}\right. \qquad (1)$$

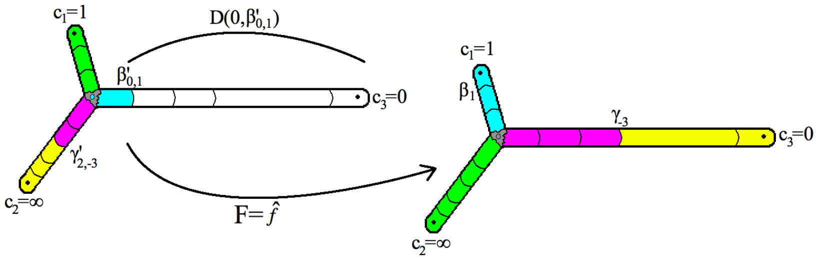
$$\text{with } M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Furthermore $\lambda(M) < 1$ implies $\exists x \in \mathbb{R}^4 \,/\, x > 0$ and $Mx < x$ □

Step 2 - The branching piece

Define

$$F_{|\widehat{\mathbb{C}}-D(0,\beta'_{0,1})} = \widehat{f}_{|\widehat{\mathbb{C}}-D(0,\beta'_{0,1})}$$

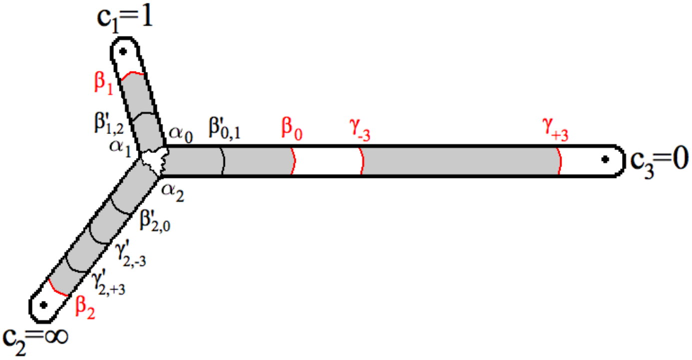


Step 3 - Preimage of the branching piece

Lemma (inverse Grötzsch's inequality - Cui Guizhen and Tan Lei)

$\exists C > 0 \,/\, \forall \beta_0, \beta_1,\ \mathrm{mod}(\beta_1, \beta_0) < \mathrm{mod}(\alpha_0, \beta_0) + \mathrm{mod}(\alpha_1, \beta_1) + C$

$(1) \Rightarrow \mathrm{mod}(\alpha_0, \beta_0) + \mathrm{mod}(\alpha_1, \beta_1) + C < \mathrm{mod}(\gamma_{-3}, \gamma_{+3})$



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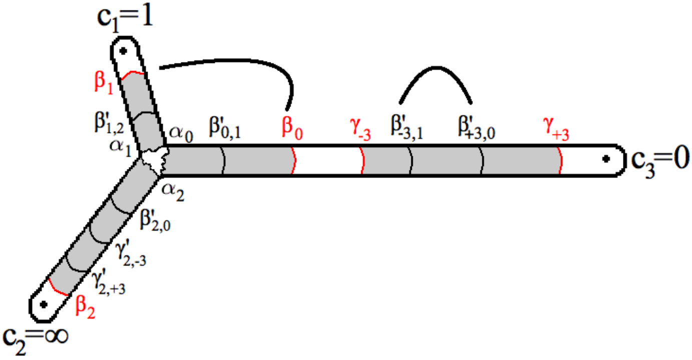
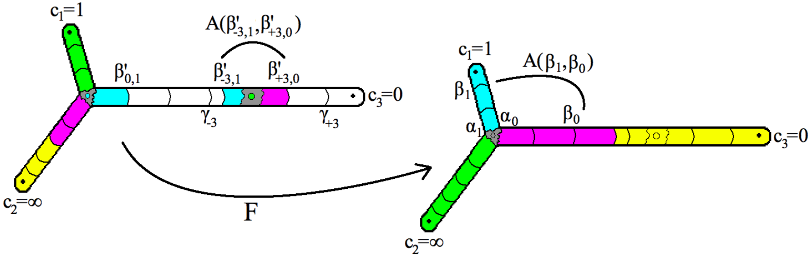
$\exists \beta'_{-3,1}, \beta'_{+3,0} \subset A(\gamma_{-3}, \gamma_{+3}) \,/\, \mathrm{mod}(\beta'_{-3,1}, \beta'_{+3,0}) = \mathrm{mod}(\beta_1, \beta_0)$

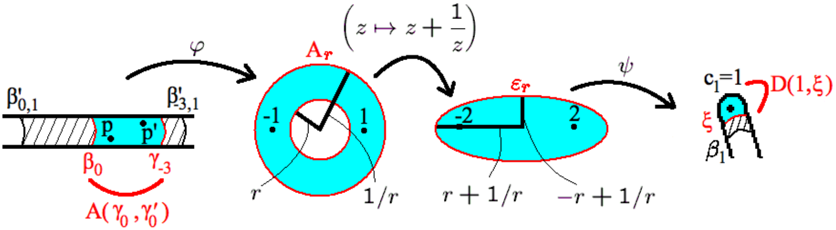


Define F on $A(\beta'_{-3,1}, \beta'_{+3,0})$ to be a biholomorphic map such that

- F maps $A(\beta'_{-3,1}, \beta'_{+3,0})$ onto $A(\beta_1, \beta_0)$
- F extends diffeomorphically to $\overline{A(\beta'_{-3,1}, \beta'_{+3,0})}$ mapping $\beta'_{-3,1}$ onto β_1 and $\beta'_{+3,0}$ onto β_0



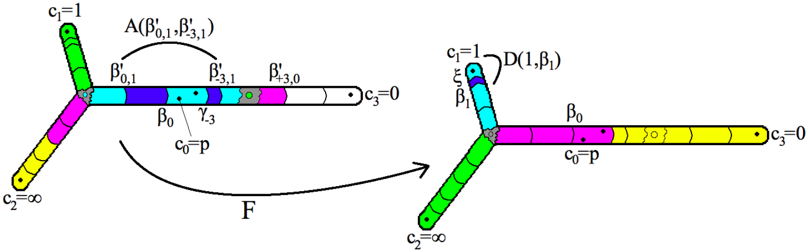
Step 4 - Folding



Define

$$F_{|A(\beta_0,\gamma_{-3})} = \psi \circ (z \mapsto z + \frac{1}{z}) \circ \varphi$$

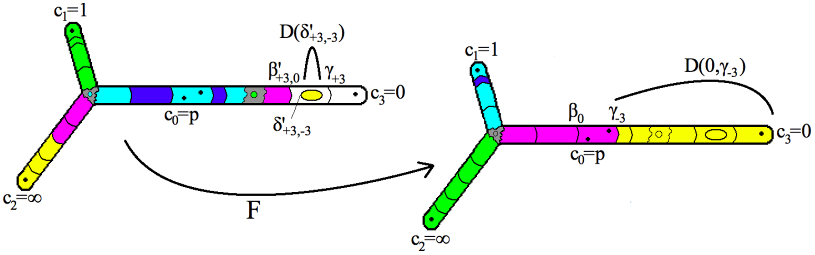
Extends quasiregularly F on $\overline{A(\beta'_{0,1}, \beta_0)} \bigcup \overline{A(\gamma_{-3}, \beta'_{-3,1})}$



Step 5 - End with an end

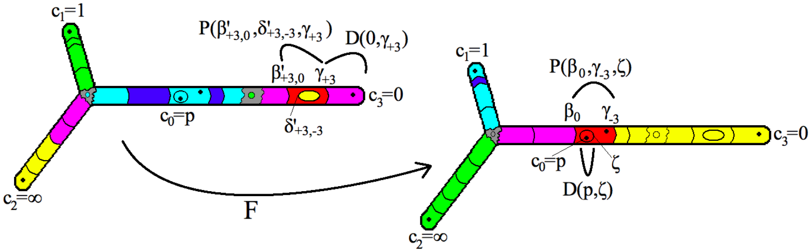
Let $\delta'_{+3,-3} \subset A(\beta'_{+3,0}, \gamma_{+3})$ be a smooth curve.
Define F on $D(\delta'_{+3,-3})$ to be a biholomorphic map such that

- F maps $D(\delta'_{+3,-3})$ onto $D(0, \gamma_{-3})$
- F extends diffeomorphically to $\overline{D(\delta'_{+3,-3})}$ mapping $\delta'_{+3,-3}$ onto γ_{-3}



Define F on $D(0, \gamma_{+3})$ to be any biholomorphic map such that

- F maps $D(0, \gamma_{+3})$ onto $D(p, \zeta) \subset A(\beta_0, \gamma_{-3})$ with $F(0) = p$
- F extends diffeomorphically to $\overline{D(0, \gamma_{+3})}$ mapping γ_{+3} onto ζ



Extends quasiregularly F on $\overline{P(\beta'_{+3,0}, \delta'_{+3,-3}, \gamma_{+3})}$

Final Step

- F is holomorphic on an open set $H \subset \widehat{\mathbb{C}}$

$$H = \underbrace{\left(\widehat{\mathbb{C}} - \overline{D(0, \beta'_{0,1})}\right)}_{\text{Step 2}} \bigcup \underbrace{A(\beta'_{-3,1}, \beta'_{+3,0})}_{\text{Step 3}} \bigcup \underbrace{A(\beta_0, \gamma_{-3})}_{\text{Step 4}}$$

$$\bigcup \underbrace{D(\delta'_{+3,-3}) \cup D(0, \gamma_{+3})}_{\text{Step 5}}$$

- F extends quasiregularly to the complement $Q = \widehat{\mathbb{C}} - H$

$$Q = \underbrace{\overline{A(\beta'_{0,1}, \beta_0)} \cup \overline{A(\gamma_{-3}, \beta'_{-3,1})}}_{\text{Step 4}} \bigcup \underbrace{\overline{P(\beta'_{+3,0}, \delta'_{+3,-3}, \gamma_{+3})}}_{\text{Step 5}}$$

- $\exists$ an open set $A \subset H$ such that $F(A) \subset A$ and $F^{\circ 2}(Q) \subset A$

$$A = A(\beta_0, \gamma_{-3}) \cup D(1, \beta_1) \cup D(\infty, \beta_2) \cup D(0, \gamma_{+3})$$

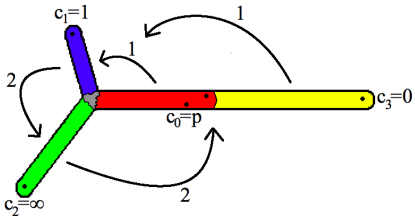
Quasiconformal surgery principle:

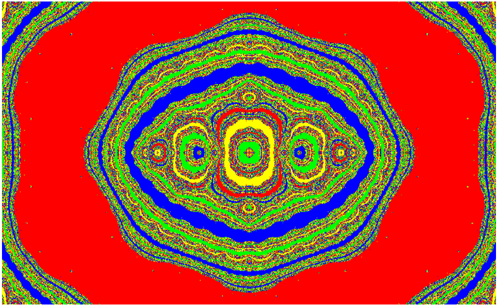
We may apply Morrey-Ahlfors-Bers theorem to get

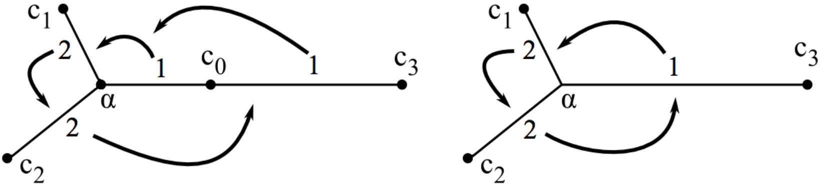
$\exists$ a quasiconformal map ϕ with F-invariant dilatation

Therefore $f = \phi \circ F \circ \phi^{-1}$ is a rational map.

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\quad F \quad} & \widehat{\mathbb{C}} \\ \phi \downarrow & & \downarrow \phi \\ \widehat{\mathbb{C}} & \xrightarrow[\quad f \quad]{} & \widehat{\mathbb{C}} \end{array}$$


c1=1
1
1
2
c3=0
c0=p
2
c2=∞





$$c_1 \xrightarrow{2} c_2 \xrightarrow{2} c_3, \quad c_3 \xrightarrow{1} c_1$$

This ramification portrait is realized by $\widehat{f} = \left(z \mapsto \dfrac{1}{(1-z)^2} \right)$

Question: Which kind of ramification portraits is realized by post-critically finite rational maps ?

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Answer: The Thurston's topological characterization !

Theorem (Thurton's topological characterization)

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ *be a ramified covering with* $|P_f| < \infty$
Then there exists a rational map $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *such that*

$\exists \varphi_0, \varphi_1$ *homeomorphisms* /

(i)
$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{\varphi_1} & \widehat{\mathbb{C}} \\ f \downarrow & & \downarrow \widehat{f} \\ \mathbb{S}^2 & \xrightarrow[\varphi_0]{} & \widehat{\mathbb{C}} \end{array}$$

(ii) $\varphi_0(P_f) = \varphi_1(P_f) = P_{\widehat{f}}$

(iii) φ_0, φ_1 *are isotopic rel. to* P_f

if and only if f *has no Thurston obstruction.*

Topological part

Definition (N-cyclic ramification portrait of polynomial type)

A ramification portrait $\mathcal{R} = (\Omega, P, \sigma, \nu)$ is N-cyclic ramification portrait of polynomial type if

- $\mathcal{R}$ is branch compatible: $\forall y \in P$, $\sum_{\sigma(x)=y} \nu(x) \leqslant \deg(\mathcal{R})$
- $\exists \infty \in \Omega \cup P / \, \sigma(\infty) = \infty$ and $\nu(\infty) = \deg(\mathcal{R})$
- $\forall \omega \in \Omega - \{\infty\}$, ω is σ-periodic
- $P - \{\infty\}$ is the union of exactly N disjoint periodic cycles

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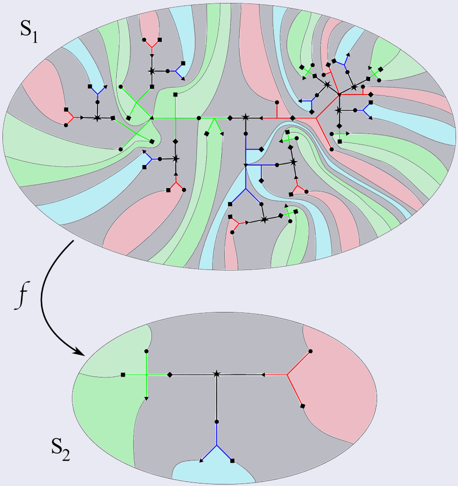
Theorem (topological realization)

Every N-cyclic ramification portrait of polynomial type is realized by a ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$.

Sketch of proof for topological realization.



□

Analytical part

Theorem (polynomial criterion)

If a topological polynomial f has a Thurston obstruction then

(i) *f has a Levy cycle Γ contained in the Thurston obstruction*

(ii) *there exist some post-critical points of f whose iterations do not accumulate a critical point*

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Corollary (Levy's criterion)

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a topological polynomial with $|P_f| < \infty$
If every critical point falls into a periodic cycle containing a critical point then f has no Thurston obstruction.

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Corollary (analytical realization)

Every N-cyclic ramification portrait of polynomial type is realized by a polynomial $\widehat{f} : \mathbb{S}^2 \to \mathbb{S}^2$.

Let $\widehat{\mathcal{R}}$ be a N-cyclic ramification portrait of polynomial type.

$$c_1^1 \xrightarrow{\widehat{\nu}(c_1^1)} c_2^1 \xrightarrow{\widehat{\nu}(c_2^1)} \cdots \xrightarrow{\widehat{\nu}(c_{n_1-1}^1)} c_{n_1}^1, \qquad c_{n_1}^1 \xrightarrow{\widehat{\nu}(c_{n_1}^1)} c_1^1$$

$$c_1^2 \xrightarrow{\widehat{\nu}(c_1^2)} c_2^2 \xrightarrow{\widehat{\nu}(c_2^2)} \cdots \xrightarrow{\widehat{\nu}(c_{n_2-1}^2)} c_{n_2}^2, \qquad c_{n_2}^2 \xrightarrow{\widehat{\nu}(c_{n_2}^2)} c_1^2$$

$$\vdots \qquad\qquad \vdots$$

$$c_1^N \xrightarrow{\widehat{\nu}(c_1^N)} c_2^N \xrightarrow{\widehat{\nu}(c_2^N)} \cdots \xrightarrow{\widehat{\nu}(c_{n_N-1}^N)} c_{n_N}^N, \qquad c_{n_N}^N \xrightarrow{\widehat{\nu}(c_{n_N}^N)} c_1^N$$

$$c_\infty \circlearrowleft \deg(\widehat{\mathcal{R}})$$

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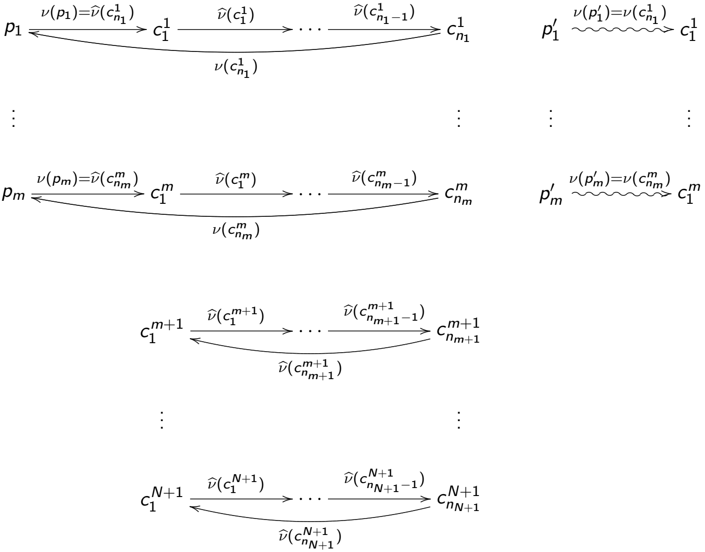
$$c_1^{N+1} \xrightarrow{\widehat{\nu}(c_1^{N+1})} c_2^{N+1} \xrightarrow{\widehat{\nu}(c_2^{N+1})} \cdots \xrightarrow{\widehat{\nu}(c_{n_{N+1}-1}^{N+1})} c_{n_{N+1}}^{N+1}, \qquad c_{n_{N+1}}^{N+1} \xrightarrow{\widehat{\nu}(c_{n_{N+1}}^{N+1})} c_1^{N+1}$$

Let $\mathcal{R}$ be the following ramification portrait.



Definition (admissible weighted Hubbard tree)

Such a ramification portrait $\mathcal{R}$ may be deduced from a weighted Hubbard tree $(\mathcal{H}, w)$ such that

- tree shape condition:
  $\mathcal{H}$ is a starlike tree around an unique branched point α, every p_i is the endpoint of two exactly two edges and every c_k^i is an end
- realization condition:
  the associated sub-ramification portrait $\widehat{\mathcal{R}}$ is a N-cyclic ramification portrait of polynomial type
- Thurston condition:
  $(\mathcal{H}, w)$ is unobstructed

Theorem (realization of admissible weighted Hubbard tree)

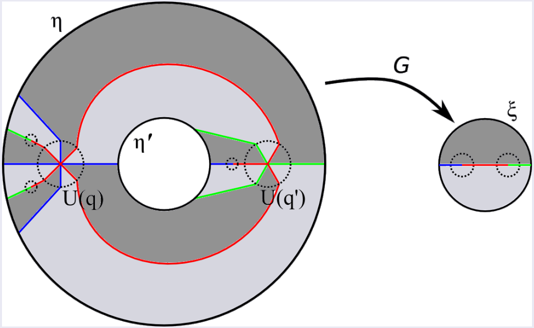
For every admissible weighted Hubbard tree $(\mathcal{H}, w)$ *there exists a rational map* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *such that*

(i) f *realizes the associated ramification portrait* $\mathcal{R}$

(ii) *the Julia set* $\mathcal{J}(f)$ *is disconnected*

Sketch of the proof.

First idea: Folding



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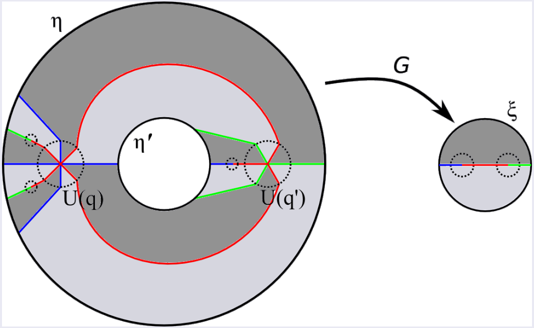
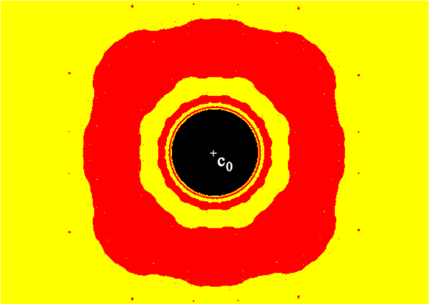


Second idea: Final Step

Use a result of Cui Guizhen and Tan Lei generalizing the Thurston's theorem for some non-post-critically finite maps. □



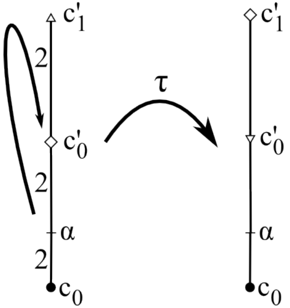
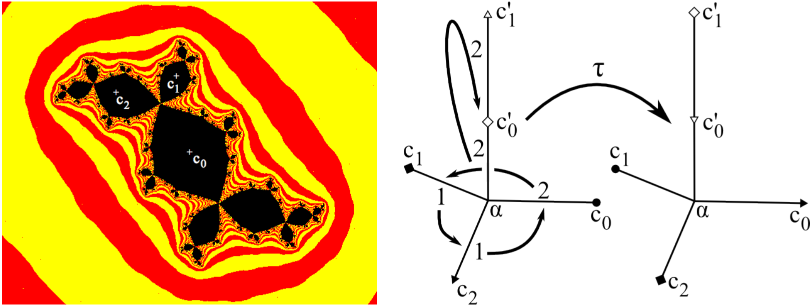


Figure: Different motifs of Persian carpets



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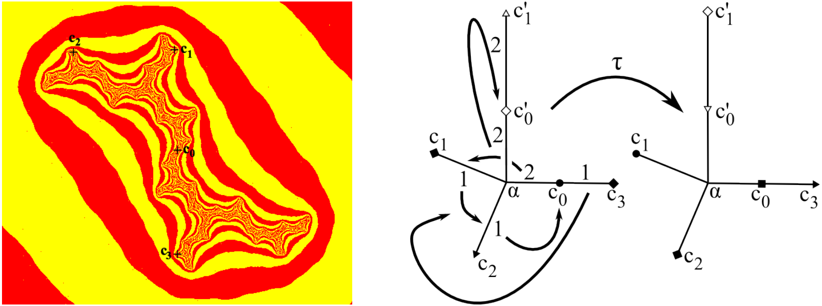

Figure: Different motifs of Persian carpets

- Enlarge the tree shape condition and the realization condition .
- Encode the exchanging dynamics of Julia components.
- Extends continuously the encoding map $\pi : \mathcal{J}_{\mathcal{H}} \to \mathcal{H}$ to $\widehat{\mathbb{C}}$.

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And more generally,

- What about the unicity ?
- What about the converse problem ?

Merci de votre attention !