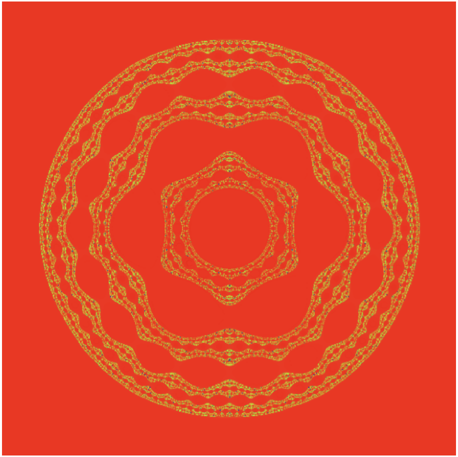
On McMullen-like mappings

Sébastien Godillon

Example 1: McMullen 1988

$$f_\lambda(z) = z^n + \frac{\lambda}{z^d} \qquad \text{where} \quad |\lambda| \ll 1 \quad \text{and} \quad \boxed{\frac{1}{n} + \frac{1}{d} < 1}$$



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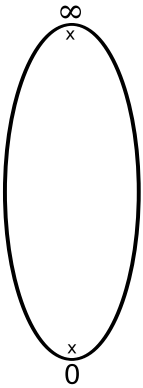

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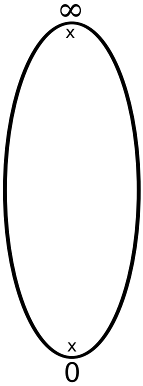

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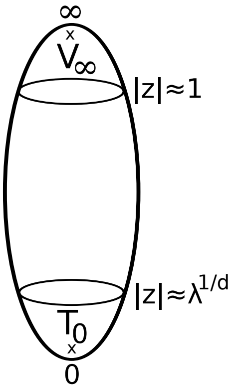
$$|\lambda| \ll 1 \Rightarrow \begin{cases} V_\infty & \approx & \{|z| > 1\} \\ T_0 & \approx & \{|z| < |\lambda|^{1/d}\} \end{cases}$$



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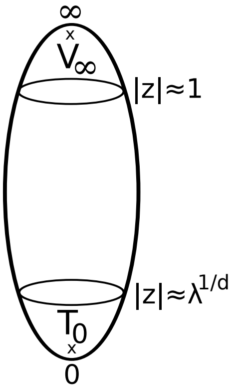

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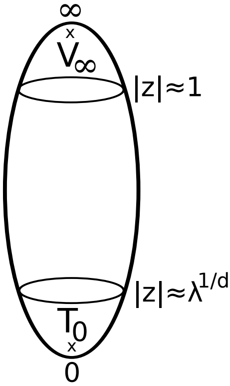

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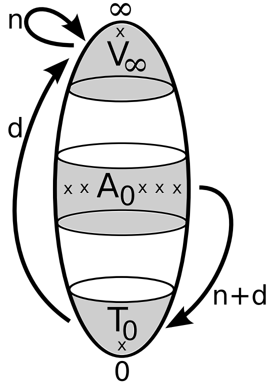

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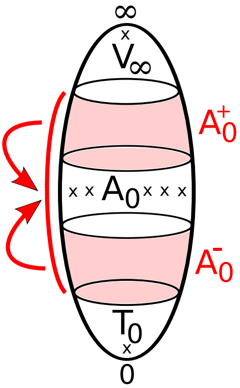

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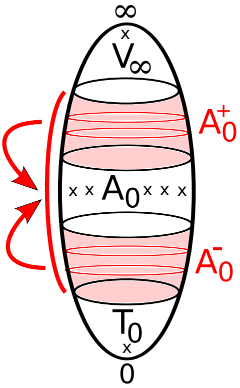

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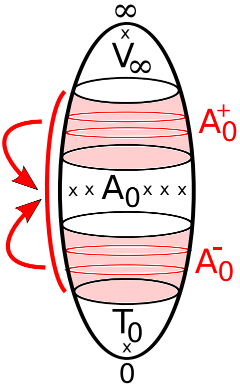
$$\mathcal{J}(f_\lambda) \simeq \Sigma_2 \times \mathbb{S}^1$$

$$\Sigma_2 := \{(\varepsilon_0, \varepsilon_1, \dots) \in \{-1, +1\}^{\mathbb{N}}\}$$

$$J \simeq \mathbb{S}^1 \leftrightarrow \varepsilon_k = \begin{cases} -1 & \text{si} \quad f_\lambda^k(J) \in A_0^- \\ +1 & \text{si} \quad f_\lambda^k(J) \in A_0^+ \end{cases}$$



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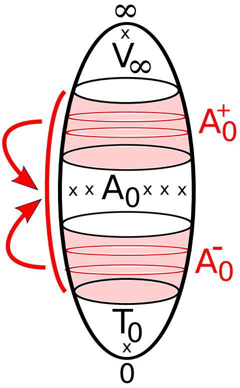
$$\text{Exchanging dynamics} \leftrightarrow \tau : \Sigma_2 \to \Sigma_2$$

$$\tau(\varepsilon_0, \varepsilon_1, \dots) := \begin{cases} (-\varepsilon_1, -\varepsilon_2, \dots) & \text{si} \quad \varepsilon_0 = -1 \\ (\varepsilon_1, \varepsilon_2, \dots) & \text{si} \quad \varepsilon_0 = +1 \end{cases}$$



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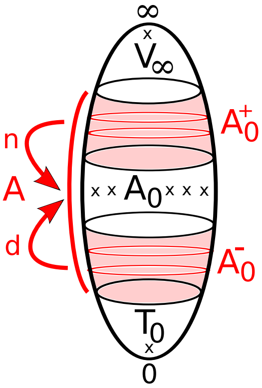

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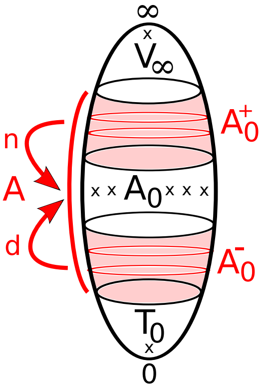
$$\Rightarrow \boxed{\frac{1}{n} + \frac{1}{d} < 1}$$



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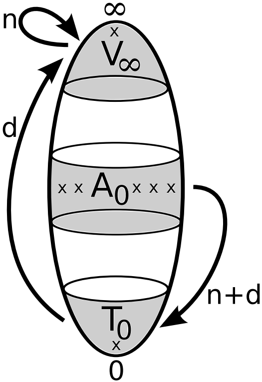
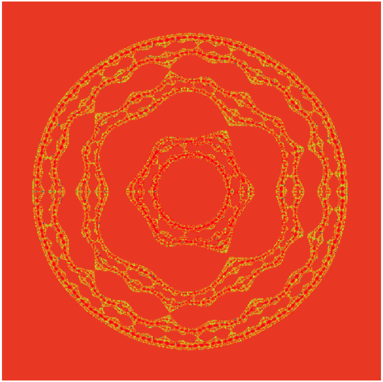
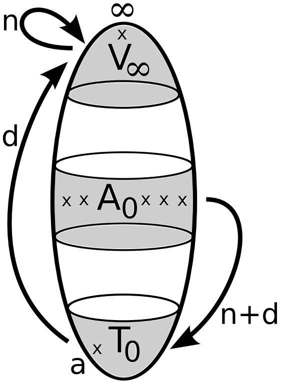


Example 2: Devaney-Marotta 2008

$$\widetilde{f}_\lambda(z) = z^n + \frac{\lambda}{(z-a)^d} \qquad \text{where} \quad |\lambda|, |a| \ll 1 \quad \text{and} \quad \boxed{\frac{1}{n} + \frac{1}{d} < 1}$$



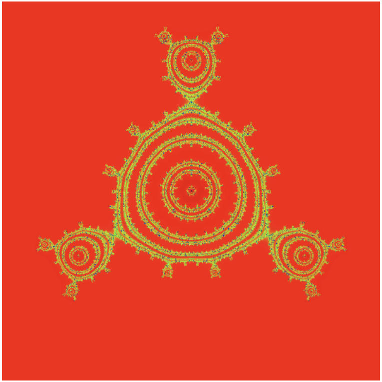
$z^3 + \frac{10^{-2}}{(z-10^{-2})^3}$



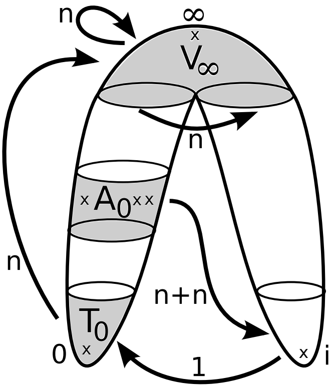
Example 3: Blanchard-Devaney-Garijo-Russell 2008

Let $c \in \mathbb{C}$ be such that
$$\begin{tikzcd} c_0 = 0 \arrow[r] & c_1 = c \arrow[r, "z^n+c"] & c_2 \arrow[r] & \ldots \arrow[r] & c_{p-1} \arrow[llll, bend right] \end{tikzcd} .$$

$$g_\lambda(z) = z^n + c + \frac{\lambda}{z^n} \qquad \text{where} \quad |\lambda| \ll 1 \quad \text{and} \quad \boxed{n \geqslant 3}$$



$z^3 + i + \frac{10^{-7}}{z^3}$

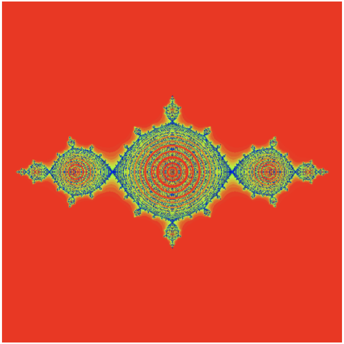


Example 4: Garijo-Marotta-Russell 2013

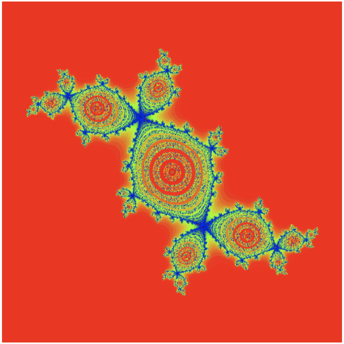
Let $c \in \mathbb{C}$ be such that

$$\begin{tikzcd} c_0 = 0 \arrow[r] & c_1 = c \arrow[r, "z^2+c"] & c_2 \arrow[r] & \dots \arrow[r] & c_{p-1} \arrow[llll, bend right] \end{tikzcd} .$$

$$h_\lambda(z) = z^2 + c + \frac{\lambda}{\prod_{j=0}^{p-1}(z-c_j)^{d_j}} \text{ where } |\lambda| \ll 1 \text{ and } \boxed{\begin{cases} 2d_1 & > & d_0 + 2 \\ d_2 & > & d_1 + 1 \\ \dots & & \\ d_0 & > & d_{p-1} + 1 \end{cases}}$$



$z^2 - 1 + \frac{10^{-22}}{z^7(z+1)^5}$



$z^2 + c_{\text{lapin}} + \frac{10^{-24}}{z^{11}(z-c_1)^7(z-c_2)^9}$

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Let $c \in \mathbb{C}$ be such that $c_0 = 0 \longrightarrow c_1 = c \xrightarrow{z^2+c} c_2 \longrightarrow \dots \longrightarrow c_{p-1}$.

$$h_\lambda(z) = z^2 + c + \frac{\lambda}{\prod_{j=0}^{p-1}(z - c_j)^{d_j}}$$ where $|\lambda| \ll 1$ and

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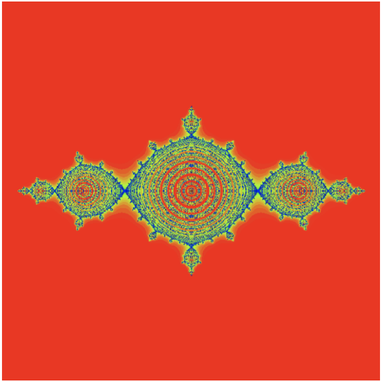
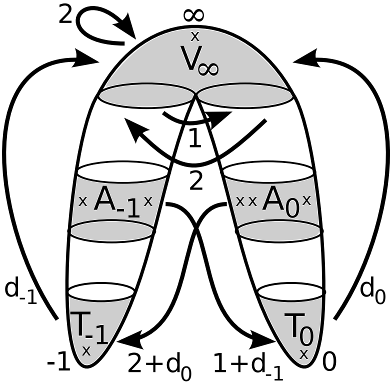
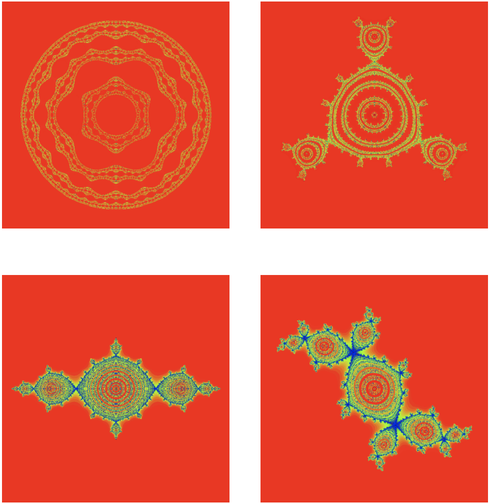




Give a global setting to the McMullen-like mappings.



$$\text{Combinatorics} \longleftrightarrow \text{Julia} \longleftrightarrow \text{Geometry}$$

$$A_0 \to T_0 \to V_\infty$$
$$f_\lambda(c_k) \in T_0$$

1. Define the McMullen-like mappings by characterizing the combinatorics with easy to check conditions.

$$\text{Combinatorics} \longleftrightarrow \text{Julia} \longleftrightarrow \text{Geometry}$$

$$A_0 \to T_0 \to V_\infty \qquad \simeq \Sigma_2 \times \mathbb{S}^1$$
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1. Define the McMullen-like mappings by characterizing the combinatorics with easy to check conditions.
2. Justify the definition by describing the Julia components and the exchanging dynamics of Julia components which are not points.

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1. Define the McMullen-like mappings by characterizing the combinatorics with easy to check conditions.
2. Justify the definition by describing the Julia components and the exchanging dynamics of Julia components which are not points.
3. Find a necessary and sufficient condition of existence which only depends on local degrees.

Let P be a post-critically finite hyperbolic polynomial.
Notations:

- $n := \deg(P)$,
- $U_\infty :=$ attractive bassin at infinity ($\partial U_\infty = \mathcal{J}(P)$),
- $N := \#\{$super-attracting periodic cycles$\}$,
- $\forall 1 \leqslant i \leqslant N,\ p_i :=$ period of the i-th cycle,
- $\{$periodic components of $\mathcal{F}(P)\} = \{U_{i,j}\ /\ 1 \leqslant i \leqslant N,\ j \in \mathbb{Z}/p_i\mathbb{Z}\}$, with $P(U_{i,j}) = U_{i,j+1}$ and $\deg(P|_{U_{i,j}} : U_{i,j} \to U_{i,j+1}) = n_{i,j}$.

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Definition 1

A pole data $\mathcal{D}$ associated to P is

- a non-empty subset of $\{U_{i,j}\ /\ 1 \leqslant i \leqslant N,\ j \in \mathbb{Z}/p_i\mathbb{Z}\}$,
- a function $U_{i,j} \mapsto d_{i,j} \in \mathbb{N} \setminus \{0\}$ defined on this subset.

Notation: $d = \deg(\mathcal{D}) := \sum_{U_{i,j} \in \mathcal{D}} d_{i,j}$.

Let $\mathcal{D}$ be a pole data associated to P.

Definition 2

$f \in \mathrm{Rat}(\widehat{\mathbb{C}})$ is a McMullen-like mapping of type $(P, \mathcal{D})$ iff

(i)

(ii)

(iii)

(iv)

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Definition 2

$f \in \mathrm{Rat}(\widehat{\mathbb{C}})$ is a McMullen-like mapping of type $(P, \mathcal{D})$ iff

(i) $\exists V_\infty \subset \widehat{\mathbb{C}}$ topological disk such that

$$f(V_\infty) = V_\infty \quad \text{and} \quad \exists \varphi \in \mathrm{Homeo}^+(\widehat{\mathbb{C}})/ \quad \begin{tikzcd} \partial U_\infty \arrow[r, "P"] \arrow[d, "\varphi"'] & \partial U_\infty \arrow[d, "\varphi"] \\ \partial V_\infty \arrow[r, "f"] & \partial V_\infty \end{tikzcd} \quad ,$$

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Notations:

- $\forall U$ component of $\mathcal{F}(f)$ disjoint from U_∞,
  $$V(U) := \text{component of } \widehat{\mathbb{C}} \setminus \varphi(\partial U) \text{ disjoint from } V_\infty,$$
- $V_{i,j} := V(U_{i,j})$.

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(ii) $\forall U \notin \mathcal{D},\ f(V(U)) = V(P(U))$,

(iii) $\forall U_{i,j} \in \mathcal{D},\ \exists \begin{cases} T_{i,j} \subset V_{i,j} \text{ topological disk} \\ A_{i,j} \Subset V_{i,j} \setminus \overline{T_{i,j}} \text{ topological annulus} \end{cases}$ such that

- $f(T_{i,j}) = V_\infty$ and $\deg(f|_{T_{i,j}}) = d_{i,j}$,
- $f|_{A_{i,j}}$ proper map and $f(A_{i,j}) =$ top. disk $\subset V(P(U_{i,j})) = V_{i,j+1}$,
- f has no critical points in $\overline{V_{i,j}} \setminus (T_{i,j} \cup A_{i,j})$,

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(ii) $\forall U \notin \mathcal{D},\ f(V(U)) = V(P(U))$,

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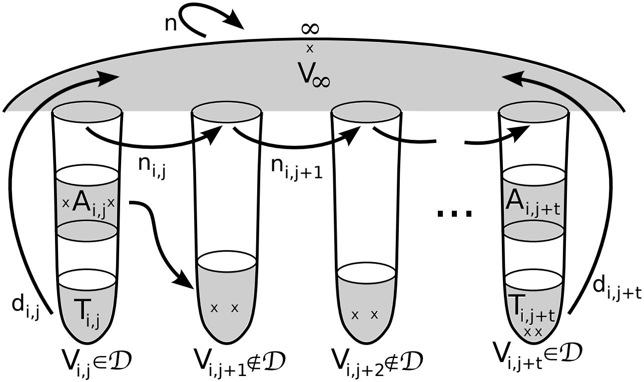
- f has no critical points in $\overline{V_{i,j}} \setminus (T_{i,j} \cup A_{i,j})$,

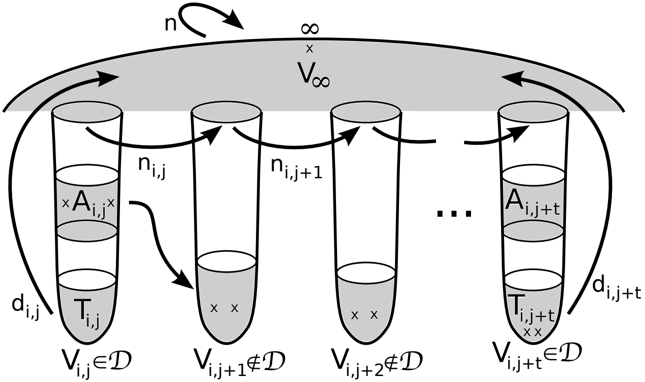
(iv) $\forall c \in \widehat{\mathbb{C}}$ critical point of f, if

$$t_c := \min\{k \geqslant 1 \ /\ \exists U_{i,j} \in \mathcal{D},\ f^k(c) \in V_{i,j}\} < +\infty$$

then $f^{t_c}(c) \in T_{i,j}$.



n
∞
x
V∞
n_{i,j}
n_{i,j+1}
A_{i,j}
A_{i,j+t}
d_{i,j}
d_{i,j+t}
T_{i,j}
T_{i,j+t}
V_{i,j}∈𝒟
V_{i,j+1}∉𝒟
V_{i,j+2}∉𝒟
V_{i,j+t}∈𝒟



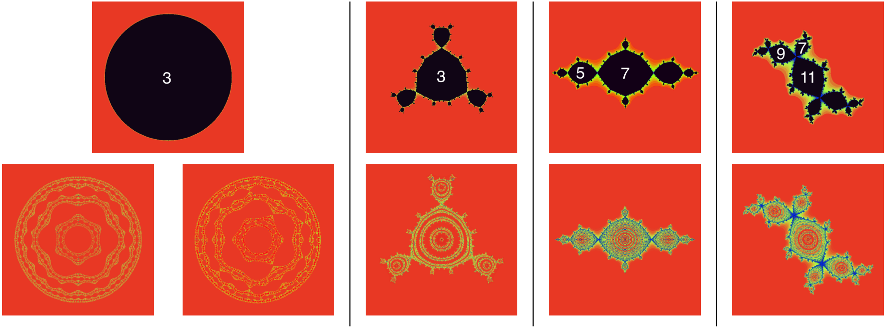
Lemma

If f is a McMullen-like mapping then

- $\forall U_{i,j} \in \mathcal{D},\ \deg(f|_{A_{i,j}} : A_{i,j} \to f(A_{i,j}) \subset V_{i,j+1}) = n_{i,j} + d_{i,j}$,
- $\forall c \in A_{i,j}$ critical point of f, $t_c = t_{i,j} := \min\{k \geqslant 1 \ / \ U_{i,j+k} \in \mathcal{D}\}$,
- we may assume $f^{t_{i,j}}(A_{i,j}) = T_{i,j+t_{i,j}}$ without loss of generality.

Known examples

$f_\lambda, \widetilde{f}_\lambda, g_\lambda, h_\lambda, \ldots$ are McMullen-like mappings.



Theorem 1 (Description of the Julia set)

If f is a McMullen-like mapping of type $(P, \mathcal{D})$ then

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2. $\mathcal{J}(f)$ contains:
   - countably many preimages of a fixed component of $\mathcal{J}(f)$ homeomorphic to $\mathcal{J}(P)$ for a certain polynomial P,
   - countably many Cantor of circles such that each circle belongs to a different component of $\mathcal{J}(f)$,
   - and if P is not affine conjugate to z^n, uncountably many point components which accumulate everywhere on $\mathcal{J}(f)$,

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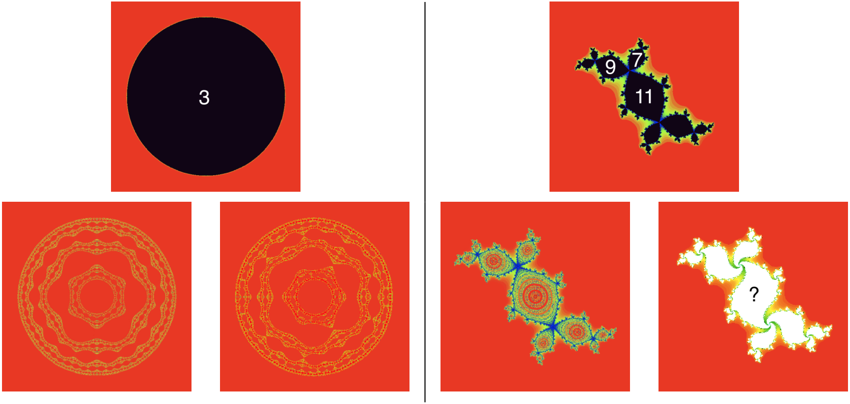
   - and if P is not affine conjugate to z^n, uncountably many point components which accumulate everywhere on $\mathcal{J}(f)$,
3. the exchanging dynamics of Julia components which are not points is encoded by a sub-shift of finite type.

Theorem 2 (Rigidity)

Two McMullen-like mappings are topologically conjugate on their Julia sets by a $\varphi \in \mathrm{Homeo}^{+}(\widehat{\mathbb{C}})$ iff they have same type $(P, \mathcal{D})$ up to affine conjugation.



Theorem 3 (Arithmetic condition of existence)

There exists a McMullen-like mapping of type $(P, \mathcal{D})$ iff

$$\boxed{\max_{1 \leqslant i \leqslant N} \left\{ \prod_{\substack{j \in \mathbb{Z}/p_i\mathbb{Z} \\ U_{i,j} \notin \mathcal{D}}} \frac{1}{n_{i,j}} \times \prod_{\substack{j \in \mathbb{Z}/p_i\mathbb{Z} \\ U_{i,j} \in \mathcal{D}}} \left(\frac{1}{n_{i,j}} + \frac{1}{d_{i,j}} \right) \right\} < 1} \qquad (\star)$$

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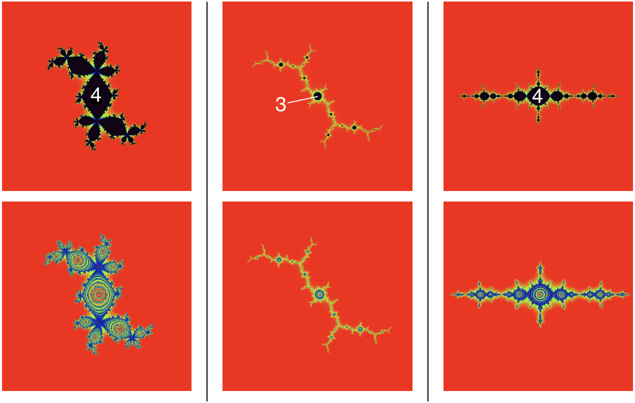
Proof: $\Rightarrow$ Thurston's obstructions theory

$\Leftarrow$ Quasi-conformal surgery

New example 1

Let $c \in \mathbb{C}$ be such that
$$\begin{tikzcd} c_0 = 0 \arrow[r] & c_1 = c \arrow[r, "z^n+c"] & c_2 \arrow[r] & \dots \arrow[r] & c_{p-1} \arrow[llll, bend right] \end{tikzcd}.$$

$$g_\lambda(z) = z^n + c + \frac{\lambda}{z^d} \qquad \text{where} \quad |\lambda| \ll 1 \quad \text{and} \quad \boxed{\frac{1}{n} + \frac{1}{d} < 1} \; (\star)$$



New example 2

Let $c \in \mathbb{C}$ be such that
$$\begin{tikzcd} c_0 = 0 \arrow[r] & c_1 = c \arrow[r, "z^2+c"] & c_2 \arrow[r] & \dots \arrow[r] & c_{p-1} \arrow[llll, bend right] \end{tikzcd}.$$

$$h_\lambda(z) = z^2 + c + \frac{\lambda}{\prod_{j=0}^{p-1}(z-c_j)^{d_j}}$$

where $|\lambda| \ll 1$ and $\boxed{\left(\frac{1}{2}+\frac{1}{d_0}\right)\left(1+\frac{1}{d_1}\right)\dots\left(1+\frac{1}{d_{p-1}}\right) < 1}$ $(\star)$

$$\boxed{\left\{\begin{array}{lcl} 2d_1 & > & d_0+2 \\ d_2 & > & d_1+1 \\ \dots & & \\ d_0 & > & d_{p-1}+1 \end{array}\right.} \Rightarrow \left\{\begin{array}{ccccc} \left(\frac{1}{2}+\frac{1}{d_0}\right) & = & \frac{d_0+2}{2d_0} & < & \frac{d_1}{d_0} \\ \left(1+\frac{1}{d_1}\right) & = & \frac{d_1+1}{d_1} & < & \frac{d_2}{d_1} \\ \dots & & & & \\ \left(1+\frac{1}{d_{p-1}}\right) & = & \frac{d_{p-1}+1}{d_{p-1}} & < & \frac{d_0}{d_{p-1}} \end{array}\right. \Rightarrow (\star)$$

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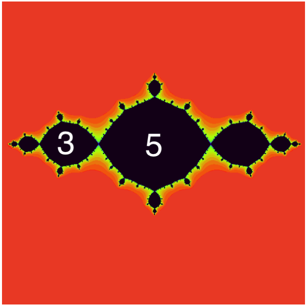
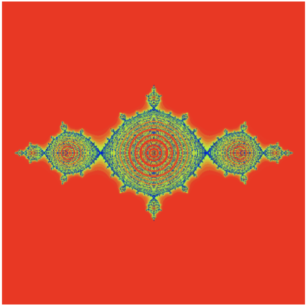




Corollary of Theorem 3

If f is a McMullen-like mapping then $\deg(f) \geqslant 4$.

Proof:

$$\boxed{\max_{1\leqslant i\leqslant N}\left\{\prod_{U_{i,j}\notin\mathcal{D}}\frac{1}{n_{i,j}}\times\prod_{U_{i,j}\in\mathcal{D}}\left(\frac{1}{n_{i,j}}+\frac{1}{d_{i,j}}\right)\right\}<1} \quad (\star)$$

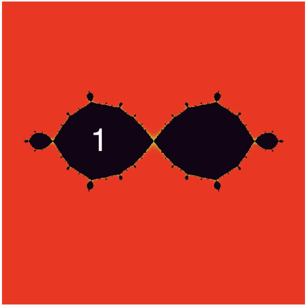
$$\Downarrow$$

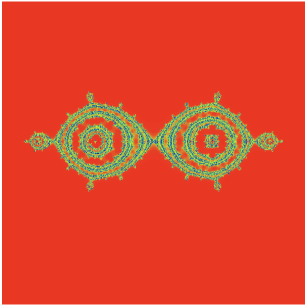
$$\deg(f) = \underbrace{\deg(P)}_{\sum_{U_{i,j}}(n_{i,j}-1)+1} + \underbrace{\deg(\mathcal{D})}_{\sum_{U_{i,j}\in\mathcal{D}} d_{i,j}} \geqslant 4$$

New example 3

$$q_\lambda(z) = 2z^3 - 3z^2 + 1 + \frac{\lambda}{z} \qquad \text{where} \quad |\lambda| \ll 1$$

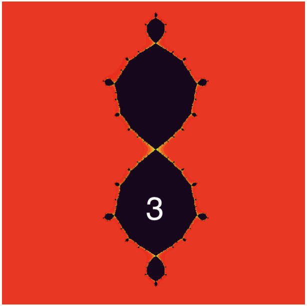
Proof: $\boxed{\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4} < 1}$ $(\star)$

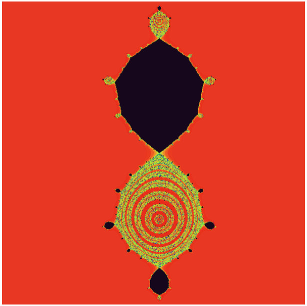


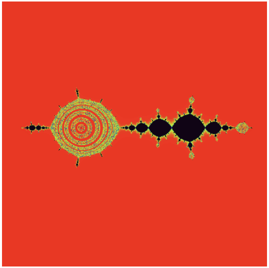
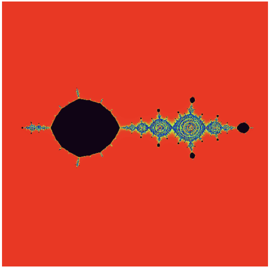
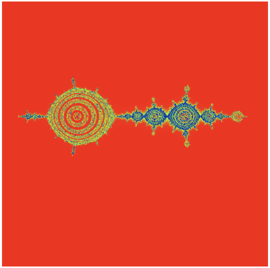


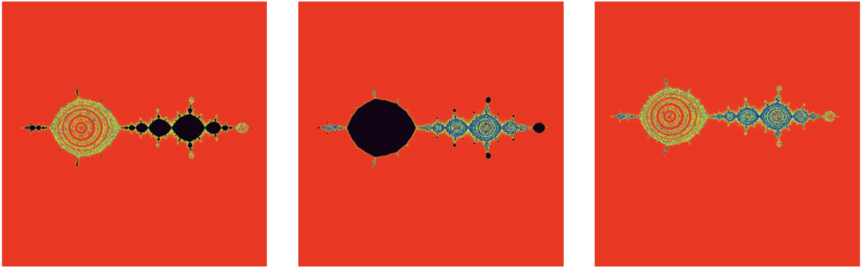
New example 4

$$r_\lambda(z) = z^3 - i\frac{3\sqrt{2}}{2}z^2 + \frac{\lambda}{z^d} \qquad \text{where} \quad |\lambda| \ll 1 \quad \text{and} \quad \boxed{\frac{1}{2} + \frac{1}{d} < 1} \; (\star)$$







Thank you for your attention!