Wandering under Bishop's trees

Sébastien Godillon

Plan

Bishop's construction 1: Motivation and main results

Bishop's construction 2: Sketch of the proof by quasiconformal foldings

Existence and non-existence of wandering domains for entire functions

Examples of wandering domains in Eremenko-Lyubich's class

Wandering under Bishop's trees

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Bishop's construction 1: Motivation and main results

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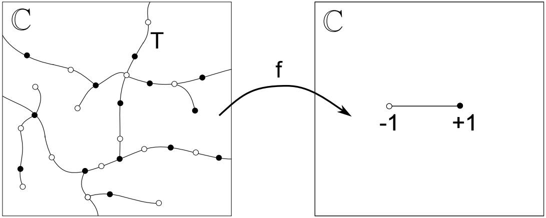
Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function with

- exactly two critical values, say -1 and $+1$
- no finite asymptotic values

Question: What does f "look like" ??

$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.



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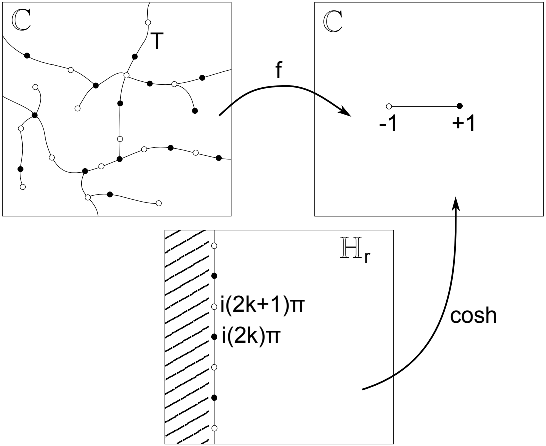
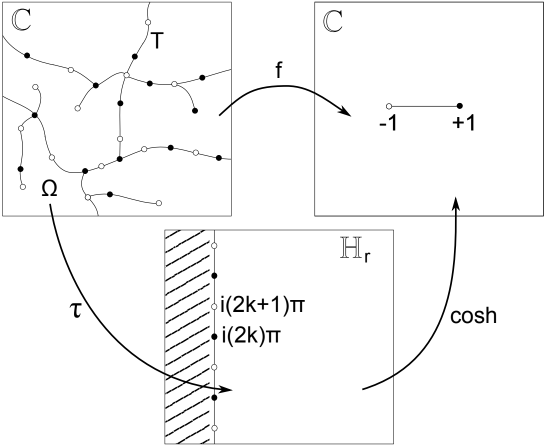


$\cosh : \mathbb{H}_r \to \mathbb{C} \backslash [-1, +1]$ is a universal cover.

$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.



$\forall \Omega$ c.c. of $\mathbb{C} \setminus T$, $\tau_{|\Omega} = (\cosh^{-1} \circ f_{|\Omega}) : \Omega \to \mathbb{H}_r$ is conformal.

Conversely: How to construct f from (T, τ) ?

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More precisely, given

- an infinite bipartite tree $T \subset \mathbb{C}$ with "smooth" enough geometry
- a map τ such that $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ is conformal, $\forall \Omega$ c.c. of $\mathbb{C} \setminus T$

does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f = \cosh \circ \tau$?

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does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f = \cosh \circ \tau$?

Main problem: $\cosh \circ \tau$ is not continuous across T in general.

Solution: Modify (T, τ) in a small neighborhood $T(r_0)$ of T.

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More precisely, replace (T, τ) by (T', η) such that

- $T \subset T' \subset T(r_0)$
- $\eta = \tau$ off $T(r_0)$
- $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is K-quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C} \setminus T'$
- $\cosh \circ \eta$ continuously extends across T'

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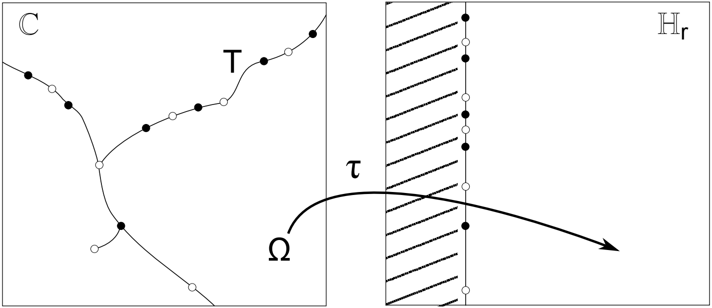
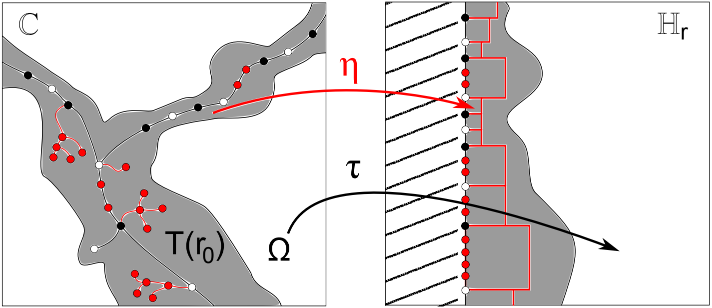
Then apply Morrey-Ahlfors-Bers measurable Riemann mapping theorem:

$\exists$ an entire function f and a quasiconformal map ϕ such that
$f \circ \phi = \cosh \circ \tau$ off $T(r_0)$



$\mathbb{C}$
T
τ
Ω
$\mathbb{H}_r$



$\mathbb{C}$
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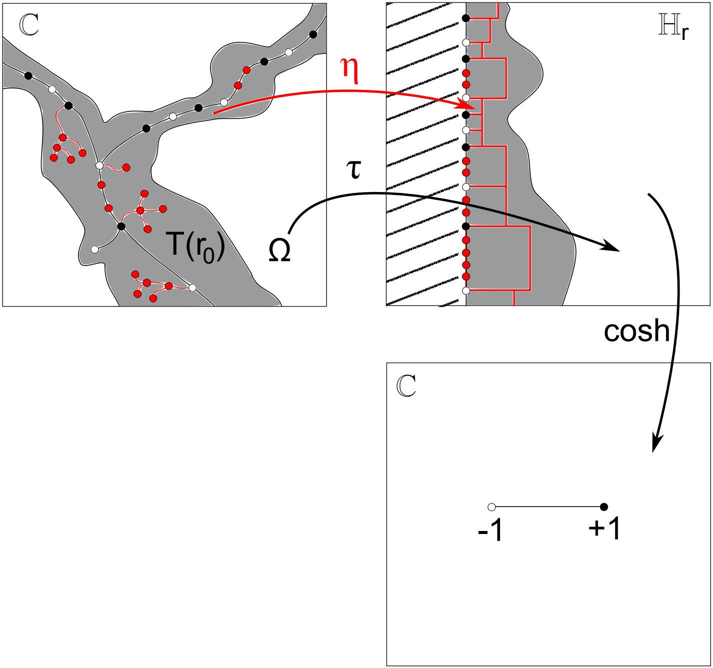

ℂ
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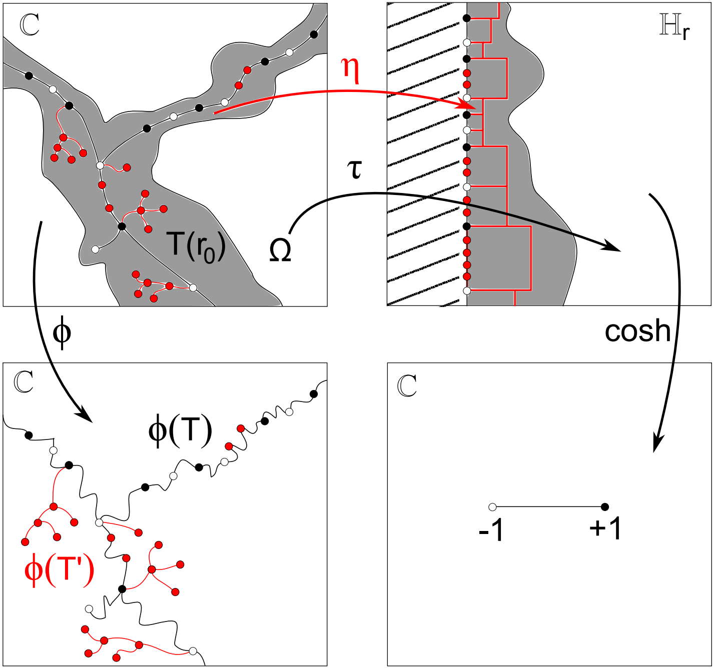

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ϕ(T')
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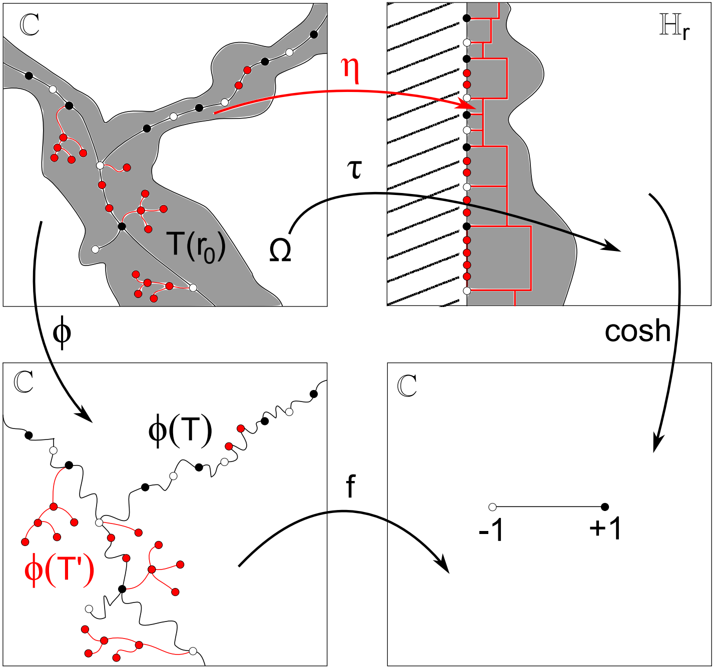

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More precisely, replace (T, τ) by (T', η) such that

- $T \subset T' \subset T(r_0)$
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- $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is K-quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C} \setminus T'$
- $\cosh \circ \eta$ continuously extends across T'

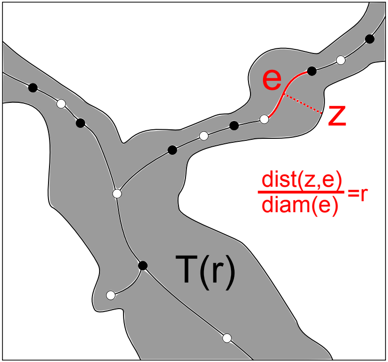
Then apply measurable Riemann mapping theorem:

$$\exists \text{ an entire function } f \text{ and a quasiconformal map } \phi \text{ such that}$$
$$f \circ \phi = \cosh \circ \tau \text{ off } T(r_0)$$

The neighborhood of T

$\forall r > 0$, define an open neighborhood of T as follows

$$T(r) = \bigcup_{e \text{ edge of } T} \left\{ z \in \mathbb{C} \ / \ \operatorname{dist}(z, e) < r \operatorname{diam}(e) \right\}$$



Lemma 0

If T has bounded geometry, namely $\exists M > 0$ such that

1. edges of T are $\mathcal{C}^2$ with uniform bounds
2. angles between adjacent edges are uniformly bounded away from 0
3. $\forall e, f$ adjacent edges, $\frac{1}{M} \leqslant \frac{\mathrm{diam}(e)}{\mathrm{diam}(f)} \leqslant M$
4. $\forall e, f$ non-adjacent edges, $\frac{\mathrm{diam}(e)}{\mathrm{dist}(e,f)} \leqslant M$

then $\exists r_0 > 0$ such that

$$\forall \Omega \text{ c.c. of } \mathbb{C} \setminus T,\ \forall \text{ square } Q \subset \mathbb{H}_r \text{ that has a } \tau_{|\Omega}\text{-edge as one side,}$$
$$Q \subset \tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$$

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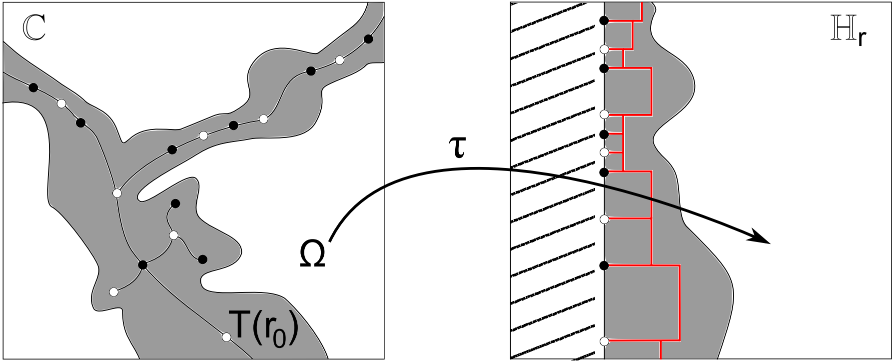

Theorem 1 (Bishop 2012)

If (T, τ) satisfies the following conditions

1. T has bounded geometry
2. every edge has τ-size $\geqslant \pi$

then $\exists$ an entire function f and a quasiconformal map ϕ such that

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Moreover

- f has exactly two critical values, -1 and $+1$
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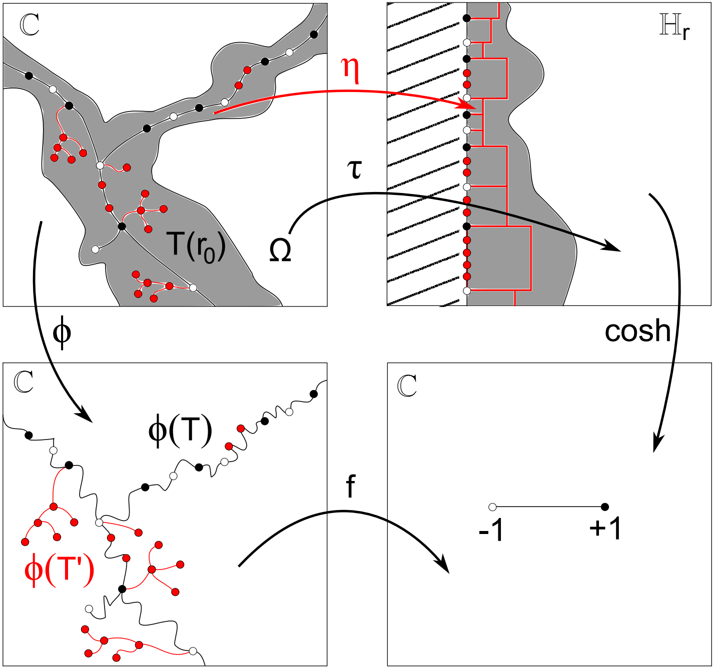
- $\phi(T) \subset f^{-1}([-1, +1]) \quad (= \phi(T'))$
- $\forall c$ critical point of f, $\deg_{\mathrm{loc}}(c, f) = \deg(c, \phi(T'))$


ℂ
T(r_0)
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Generalization: Can we construct f with

- more critical values than only -1 and $+1$?
- some finite asymptotic values ?
- arbitrary high degree critical points ?

Solution: Let T be an infinite bipartite **graph**.

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The c.c. of $\mathbb{C}\setminus T$ are sorted in three different types:

R-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ conformally

D-components: $\tau_{|\Omega} : \Omega \to \mathbb{D}$ conformally

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More precisely:

R	$\Omega \xrightarrow{\tau_{\|\Omega}} \mathbb{H}_r \xrightarrow{\cosh} \mathbb{C} \setminus [-1, +1]$
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D	$(\Omega,\star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{z\mapsto z^{d_\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{\rho_\Omega}$	$(\mathbb{D},w_\Omega)$
L	(Ω,∞)	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{H}_\ell,-\infty)$	$\xrightarrow{\exp}$	$(\mathbb{D},0)$	$\xrightarrow{\rho_\Omega}$	$(\mathbb{D},v_\Omega)$

where $\rho_\Omega : \mathbb{D} \to \mathbb{D}$ is quasiconformal with $\rho_\Omega(z) = z$, $\forall z \in \partial\mathbb{D}$.

Theorem 2 (Bishop 2012)

If (T, τ) satisfies the following conditions

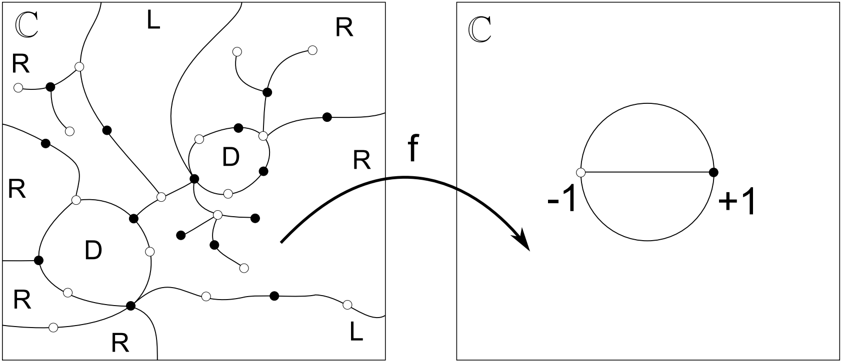
1. T has bounded geometry
2. on R-components, every edge has τ-size $\geqslant \pi$
3. D,L-components only share edges with R-components

then $\forall\ (d_\Omega \geqslant 2, w_\Omega \in \frac{3}{4}\mathbb{D})_{\Omega\in\{\text{D-components}\}}$ and $(v_\Omega \in \frac{3}{4}\mathbb{D})_{\Omega\in\{\text{L-components}\}}$,
$\exists$ an entire function f and a quasiconformal map ϕ such that

$$f \circ \phi = \sigma \circ \tau \text{ off } T(r_0) \quad \text{with } \sigma(z) = \begin{cases} \cosh(z) & \text{on R-components} \\ \rho_\Omega(z^{d_\Omega}) & \text{on D-components} \\ \rho_\Omega(\exp(z)) & \text{on L-components} \end{cases}$$

Moreover

- quasiconformal foldings only occur in R-components
- the only critical values of f are ± 1 and $(w_\Omega)_{\Omega\in\{\text{D-components}\}}$
- the only asymptotic values of f are $(v_\Omega)_{\Omega\in\{\text{L-components}\}}$
- $\forall$ D-component Ω, $\exists c \in \phi(\Omega)$ crit. point of f with $\deg_{\mathrm{loc}}(c, f) = d_\Omega$



ℂ
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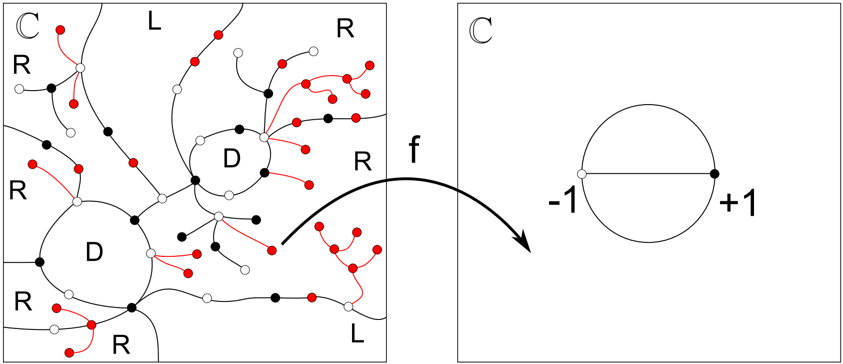

ℂ
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+1

Corollary (Bishop 2012)

Let $E, F \subset \mathbb{C}$ be two bounded countable sets with $\mathrm{card}(E) \geqslant 2$.
Then $\exists$ an entire function f such that

- E is the set of critical values of f
- F is the set of asymptotic values of f

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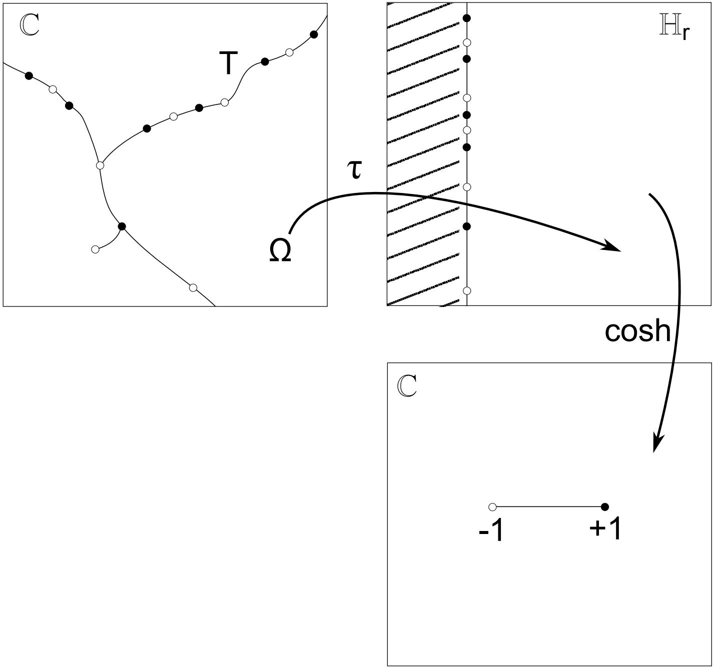
Bishop's construction 2: Sketch of the proof by quasiconformal foldings

Sébastien Godillon



$\mathbb{C}$
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cosh
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-1
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Theorem 1 (Bishop 2012)

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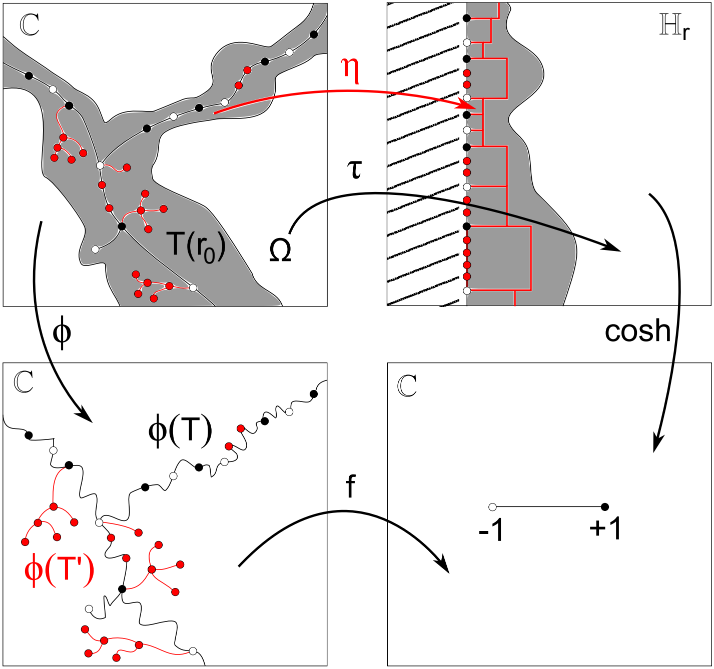

ℂ
T(r0)
η
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Idea of the proof: Construct (T', η) such that

- $T \subset T' \subset T(r_0)$
- $\eta = \tau$ off $T(r_0)$
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Main problem: the behavior of cosh on the two τ-edges of e, $\forall$ egde e.

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Main problem: the behavior of cosh on the two τ-edges of e, $\forall$ egde e.

More precisely, $\forall n \in \mathbb{Z}$, $\cosh : i\pi[n, n+1] \xrightarrow{\text{homeo}} [-1, +1]$
but

1. the two τ-edges of e are not of the form $i\pi[n, n+1]$ in general
2. the two τ-edges of e have different size in general (but $\geqslant \pi$)

Particular case: $\forall$ edge e, the two τ-edges of e have same size $\geqslant \pi$.

Lemma 1

$\exists K \geqslant 1$ such that
$\forall \Omega$ c.c. of $\mathbb{C}\setminus T$, $\exists$ a map $(\lambda_\Omega \circ \imath_\Omega) : \tau_{|\Omega}(\Omega) = \mathbb{H}_r \to \mathbb{H}_r$ such that

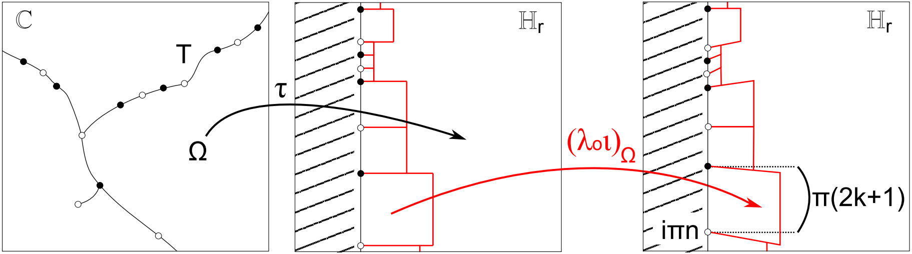
(i) $(\lambda_\Omega \circ \imath_\Omega) = \mathrm{Id}$ off $\tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$

(ii) $(\lambda_\Omega \circ \imath_\Omega) : \tau_{|\Omega}(\Omega) = \mathbb{H}_r \to \mathbb{H}_r$ is K-quasiconformal

(iii) $\forall$ edge $e \subset \partial\Omega_1 \cap \partial\Omega_2$, $(\lambda_{\Omega_j} \circ \imath_{\Omega_j}) \circ \tau_{|\Omega_j}$ continuously extends to e with

$$\begin{cases} \Big((\lambda_{\Omega_j} \circ \imath_{\Omega_j}) \circ \tau_{|\Omega_j}\Big)(e) = i\pi[n_j, n_j + (2k+1)] \quad \text{with } n_j \in \mathbb{Z}, k \in \mathbb{N} \\ (\lambda_{\Omega_1} \circ \imath_{\Omega_1}) \circ \tau_{|\Omega_1} - (\lambda_{\Omega_2} \circ \imath_{\Omega_2}) \circ \tau_{|\Omega_2} = i\pi(n_1 - n_2) \in i\pi 2\mathbb{Z} \quad \text{on } e \end{cases}$$

$$\begin{cases} \imath_\Omega : \mathbb{H}_r \to \mathbb{H}_r \text{ moves the vertices into } i\pi\mathbb{Z} \\ \lambda_\Omega : \mathbb{H}_r \to \mathbb{H}_r \text{ fixes } i\pi\mathbb{Z} \text{ and makes the continuity across } T \end{cases}$$



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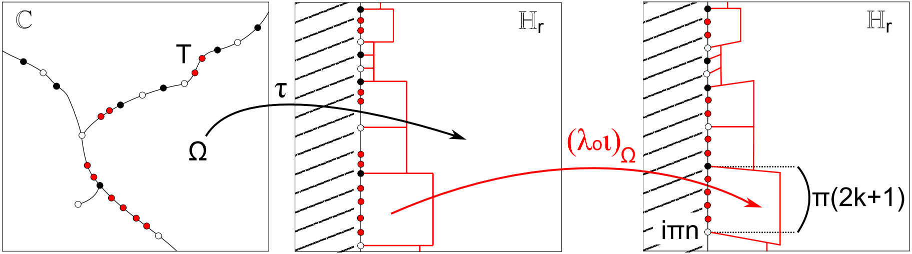

$\cosh(i\pi\mathbb{Z}) = \{-1, +1\}$ leads to extra vertices.

Particular case: $\forall$ edge e, the two τ-edges of e have same size $\geqslant \pi$.

Using Lemma 1, define

$$\begin{cases} \eta_{|\Omega} = (\lambda_\Omega \circ \imath_\Omega) \circ \tau_{|\Omega}, \ \forall \Omega \text{ c.c. of } \mathbb{C} \setminus T \\ T' = T \text{ with extra vertices coming from } \eta^{-1}(i\pi\mathbb{Z}) \end{cases}$$

then

$$\begin{array}{rcl} \text{(i)} & \Longrightarrow & \eta = \tau \text{ off } T(r_0) \\ \text{(ii)} & \Longrightarrow & \eta_{|\Omega'} : \Omega' \to \mathbb{H}_r \text{ is } K\text{-quasiconformal, } \forall \Omega' \text{ c.c. of } \mathbb{C} \setminus T' \\ \text{(iii)} & \Longrightarrow & \cosh \circ \eta \text{ continuously extends across } T' \end{array}$$

$\square$

General case: by proceeding as for the particular case, we may assume that

$$\forall \text{ edge } e \subset \partial\Omega_1 \cap \partial\Omega_2,\ \tau_{|\Omega_j} \text{ continuously extends to } e \text{ with}$$
$$\tau_{|\Omega_j}(e) = i\pi[n_j, n_j + (2k_j + 1)] \quad \text{with } n_j \in \mathbb{Z}, k_j \in \mathbb{N}$$

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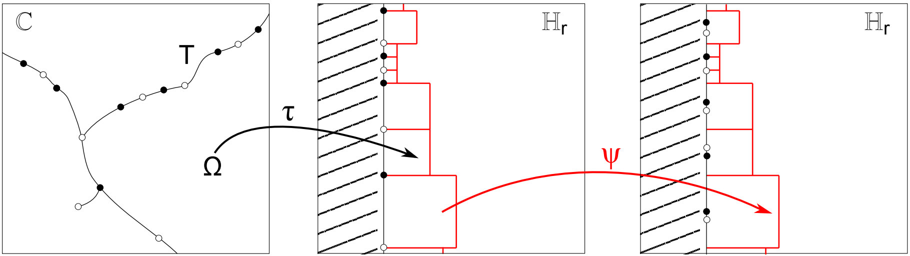

Problem: Find a quasiconformal map ψ from a square to itself such that

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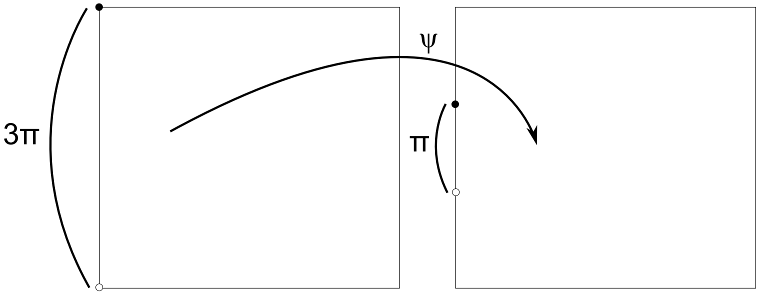

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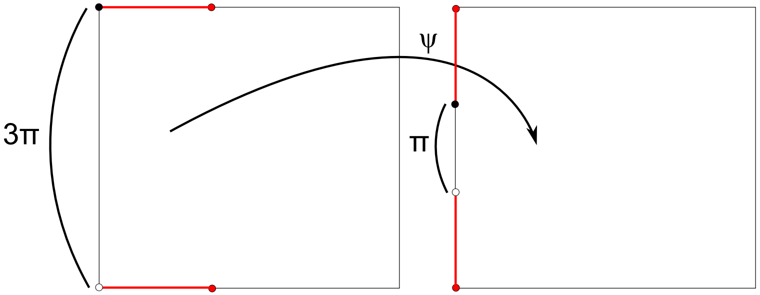

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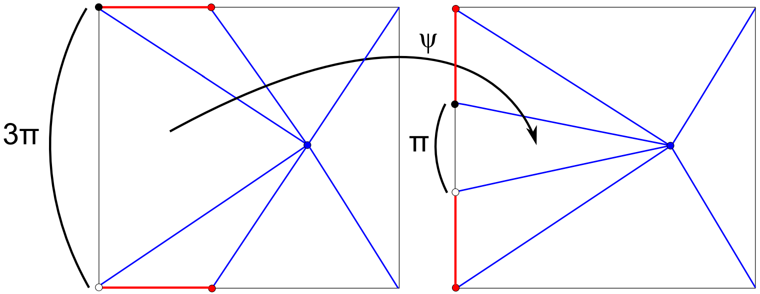

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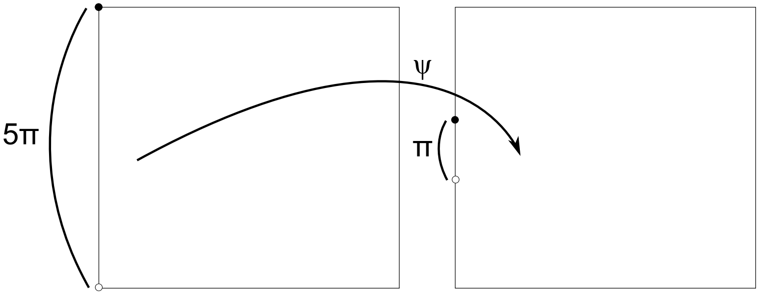

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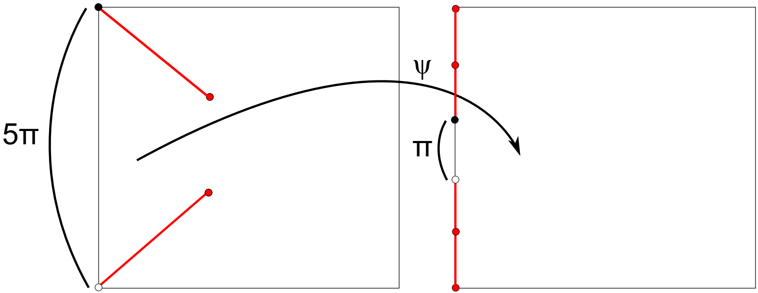

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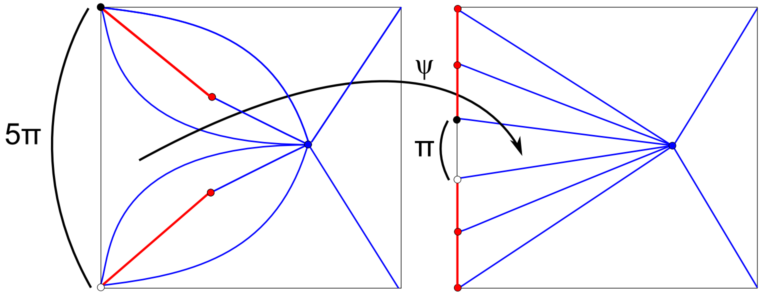

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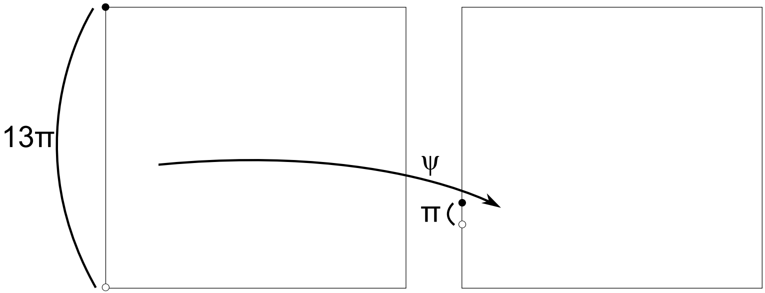

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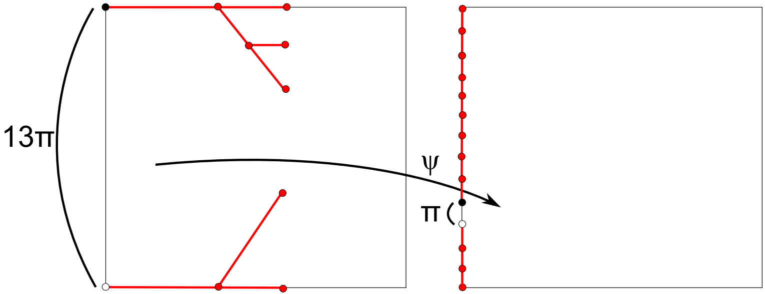

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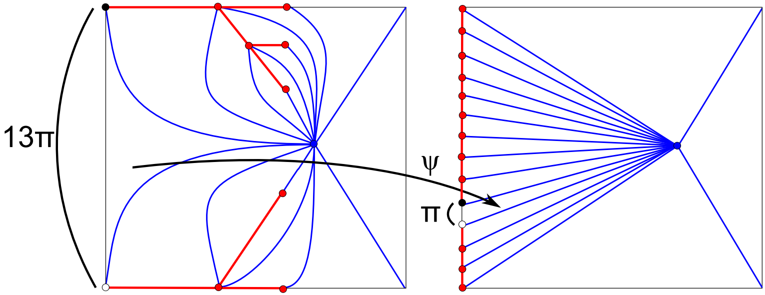

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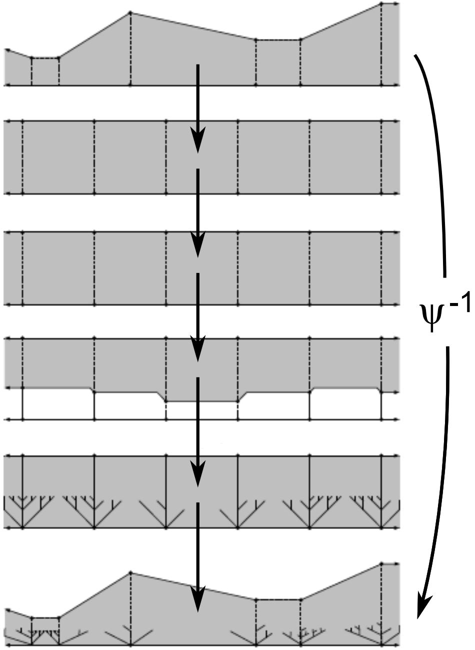
BUT

the dilatation of ψ should be uniformly bounded independently of the square size.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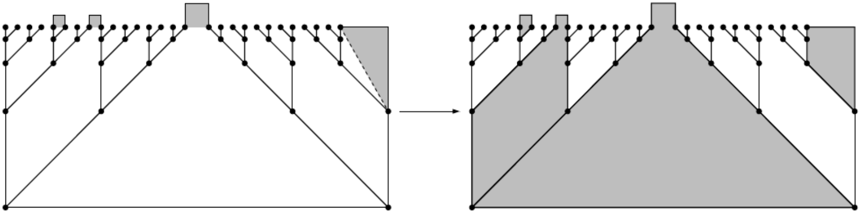

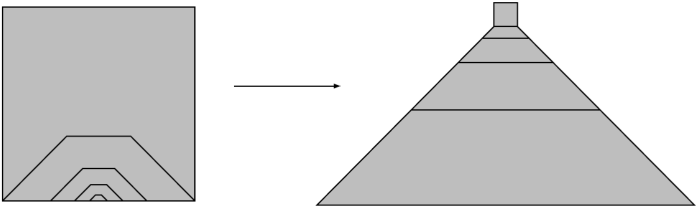

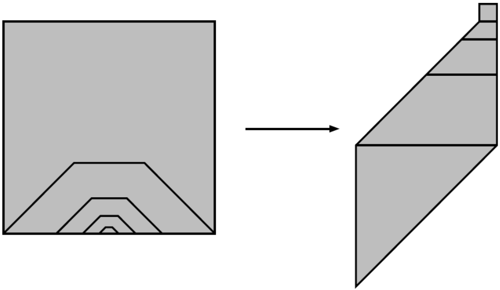

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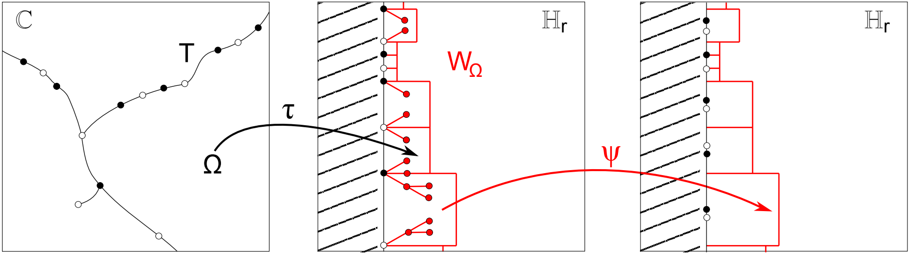

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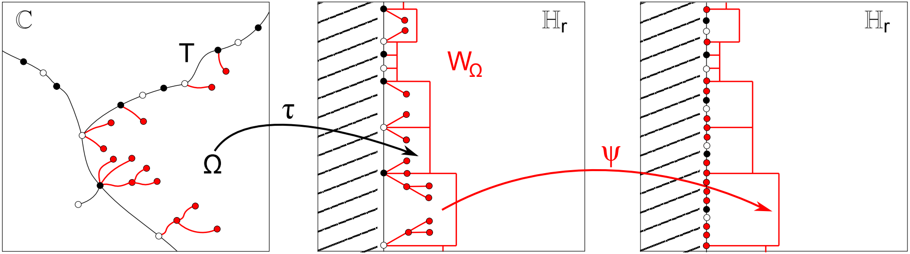

$\tau_{|\Omega}^{-1}(\partial W_\Omega)$ leads to extra vertices and edges.

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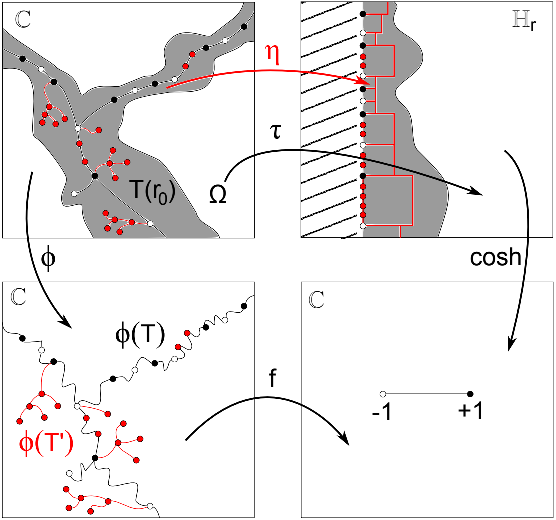
Using Lemma 2, define

$$\begin{cases} \eta_{|\Omega} = \psi_\Omega \circ \tau_{|\Omega},\ \forall \Omega \text{ c.c. of } \mathbb{C}\setminus T \\ T' = T \text{ with extra vertices and edges coming from } \eta^{-1}(\partial\mathbb{H}_r) \end{cases}$$

then

$$\begin{aligned} \text{(i)} \quad &\implies \quad \eta = \tau \text{ off } T(r_0) \\ \text{(ii)} \quad &\implies \quad \eta_{|\Omega'} : \Omega' \to \mathbb{H}_r \text{ is } K\text{-quasiconformal},\ \forall \Omega' \text{ c.c. of } \mathbb{C}\setminus T' \\ \text{(iii)} \quad &\implies \quad \cosh \circ \eta \text{ continuously extends across } T' \end{aligned}$$

$\square$



ℂ
η
$\mathbb{H}_r$
τ
T(r0)
Ω
ϕ
cosh
ℂ
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ℂ
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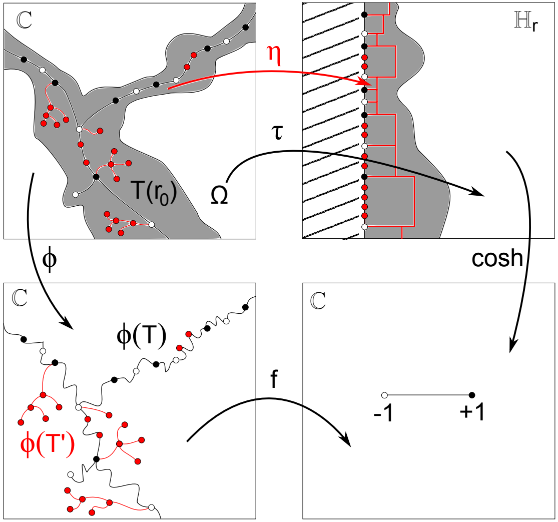
-1
+1
ϕ(T')



Tak for din opmærksomhed !

Wandering under Bishop's trees

-

Existence and non-existence of wandering domains for entire functions

Sébastien Godillon

Wandering domain

Let f be a rational map or a transcendental entire function.
A Fatou domain U of f is said to be wandering if

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If f is a rational map then f has no wandering domains.

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Singular set

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function.
Denote by $\mathcal{S}(f) = \overline{\mathrm{Crit}(f) \cup \mathrm{Asym}(f)}$ the set of finite **singular values**.

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Eremenko-Lyubich, Goldberg-Keen, 1986

If $|\mathcal{S}(f)| < +\infty$ then f has no wandering domains.

Main tools: quasiconformal deformations

Baker, 1975

If U is a multiply connected Fatou domain of f then U is wandering.

Baker, 1976

$$g(z) = \frac{1}{4e} z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{\gamma_n}\right) \quad \text{for suitable } \gamma_n > 1$$

has (multiply connected and hence) wandering domains.

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$$\begin{cases} f_1(z) = z - 1 + e^{-z} + 2\pi i \\ f_2(z) = z + \frac{e^{2\pi i \alpha} - 1}{2\pi} \sin(2\pi z) + 1 \quad \text{for suitable } \alpha \in \mathbb{R} \end{cases}$$

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(Devaney et al., 1989) ??

$$f_3(z) = z + 2\pi \sin(z)$$

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Fatou, 1920

If U is wandering then every limit function of $\{f^n|_U\}_{n\geqslant 1}$ is constant.

In particular, U is either:

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oscillating:	$\exists (n_k, m_k)$,	$f^{n_k}|_U \xrightarrow[k\to+\infty]{} \infty$ and $f^{m_k}|_U \xrightarrow[k\to+\infty]{} a \in \mathcal{J}(f)$
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If $f^{n_k}|_U \xrightarrow[k\to+\infty]{} a$ then $a \in \overline{E} \cup \{\infty\}$ where $E = \bigcup_{s\in\mathcal{S}(f)} \bigcup_{n\geqslant 1} f^n(s)$.

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Bergweiler et al., 1993

If $f^{n_k}|_U \xrightarrow[k\to+\infty]{} a$ then $a \in E' \cup \{\infty\}$.

$z \mapsto \exp(z)$, $z \mapsto \frac{\sin(z)}{z}$, and $z \mapsto \frac{\pi^2}{\pi^2 - z^2}\sin(z)$ have no wandering domains.

Eremenko-Lyubich, 1987

$\exists$ an entire function f which has (Fatou domains with infinitely many finite constant limit functions and hence) oscillating wandering domains.

Main tool: approximation theory

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Singh, 2003

$\exists$ two entire functions f, g and a domain $U \subset \mathbb{C}$ which lies in

$$\begin{cases} \text{a periodic domain} & \text{for} \quad f,\ g, \text{ and } g \circ f \\ \text{a wandering domain} & \text{for} \quad f \circ g \end{cases}$$

However $g \circ f$ must have wandering domains (Bergweiler-Wang, 1998).

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Question: Do there exist f, g which both have no wandering domains but whose composition $f \circ g$ has some ?

Eremenko-Lyubich's class

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Eremenko-Lyubich, 1992

If $f \in \mathcal{B}$ then any wandering domain is either oscillating or bounded.

Main result: $\mathcal{I}(f) \subset \mathcal{J}(f)$ (and hence $\mathcal{J}(f) = \overline{\mathcal{I}(f)}$)

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Mihaljević-Rempe, 2012

If $f \in \mathcal{B}$ satisfies $\sup_{s \in \mathcal{S}(f)} |f^n(s)| \xrightarrow[n \to +\infty]{} +\infty$ and a certain condition $(\star)$ then f has no wandering domains.

Main tool: hyperbolic geometry

$z \mapsto \lambda \frac{\sinh(z)}{z} + a$ for every $\lambda, a \in \mathbb{R}$ has no wandering domains.

Bishop, 2012

$\exists$ an entire function $f \in \mathcal{B}$ which has (oscillating) wandering domains (with infinitely many finite constant limit functions).

Main tool: Bishop's construction by quasiconformal foldings

Wandering under Bishop's trees

-

Examples of wandering domains in Eremenko-Lyubich's class

Sébastien Godillon

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Bishop's example is of the form:

$$f = F \circ \phi \quad \text{with} \quad \begin{cases} F : \mathbb{C} \to \mathbb{C} \text{ quasiregular (transcendental)} \\ \phi : \mathbb{C} \to \mathbb{C} \text{ quasiconformal so that } \mu_{\phi^{-1}} = F^*(\mu_0) \end{cases}$$

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- $\forall z \in \mathbb{C}$, $F(-z) = F(z)$ and $F(\bar{z}) = \overline{F(z)}$
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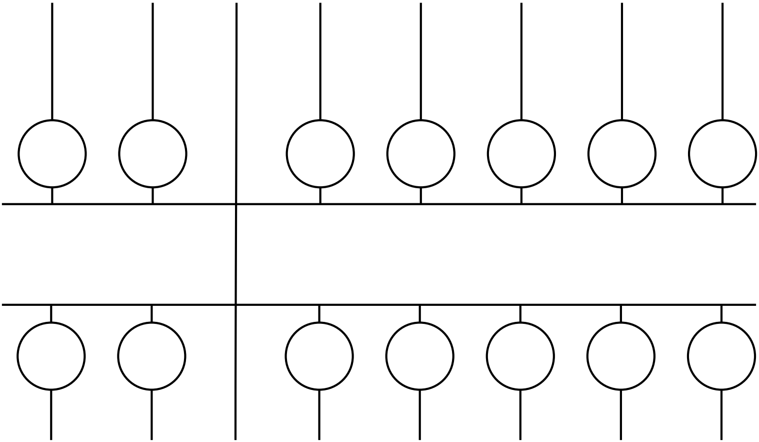
- $\mathrm{supp}\,(F^*(\mu_0))$ is small enough in order that we may find $\phi \approx \mathrm{Id}_{\mathbb{C}}$

F is constructed following an infinite graph.



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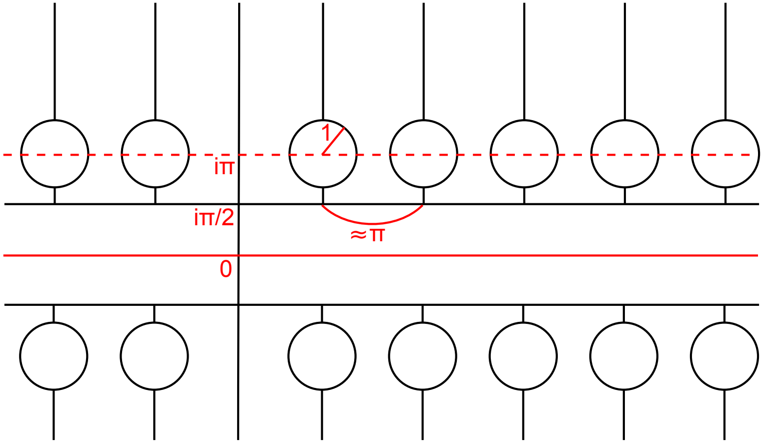
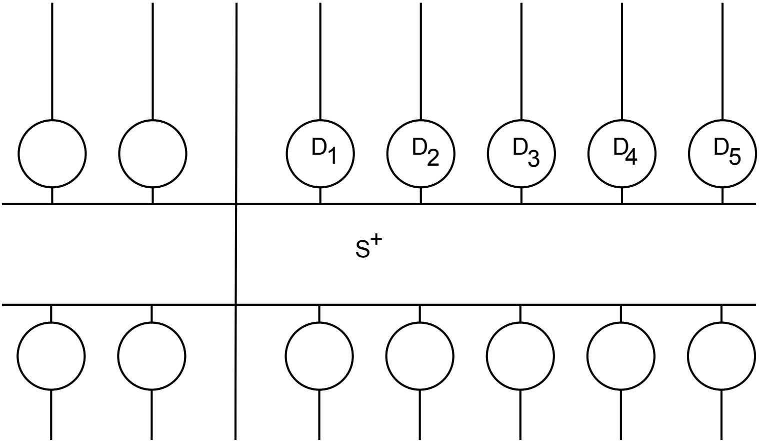
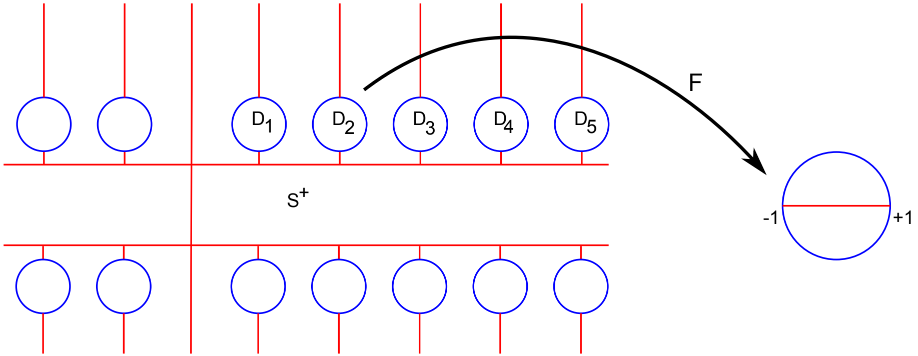


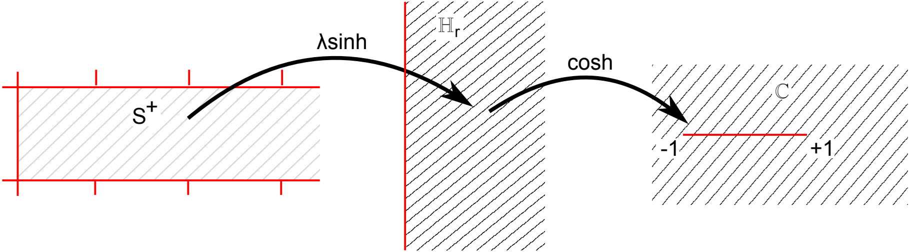
F is constructed following an infinite graph.



$$F \text{ maps } \begin{cases} \text{straight lines} & \text{onto} \quad [-1,+1] \\ \text{circle arcs} & \text{onto} \quad \partial\mathbb{D} \end{cases}$$



$$
\begin{array}{rccccc}
F: & S^+ & \xrightarrow{\lambda\,\mathsf{sinh}} & \mathbb{H}_r & \xrightarrow{\mathsf{cosh}} & \mathbb{C}\backslash[-1,+1] \\
 & z & \longmapsto & & & \mathsf{cosh}(\lambda\,\mathsf{sinh}(z))
\end{array}
$$



for some parameter $\boxed{\lambda > 0}$.

$$
\begin{array}{ccccc}
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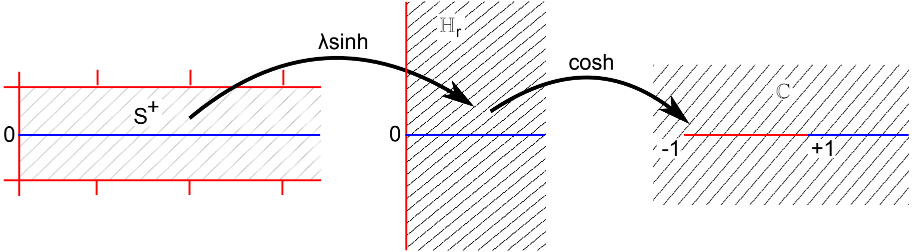
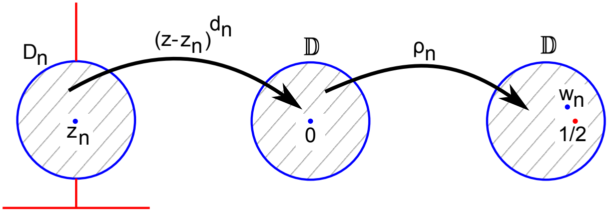


for some parameter $\boxed{\lambda > 0}$.

For every $n \geqslant 1$,

$$\begin{array}{cccccc} F: & (D_n, z_n) & \xrightarrow{z \mapsto (z - z_n)^{d_n}} & (\mathbb{D}, 0) & \xrightarrow{\rho_n} & (\mathbb{D}, w_n) \\ & z & \longmapsto & & & \rho_n\left((z - z_n)^{d_n}\right) \end{array}$$

$$\text{with} \begin{cases} \rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal} \\ \rho_n(0) = w_n \end{cases}$$



for some parameters $\boxed{d_n \xrightarrow[n \to +\infty]{} +\infty}$ and $\boxed{w_n \xrightarrow[n \to \infty]{} \frac{1}{2}}$.

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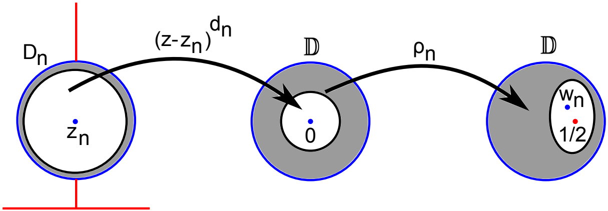
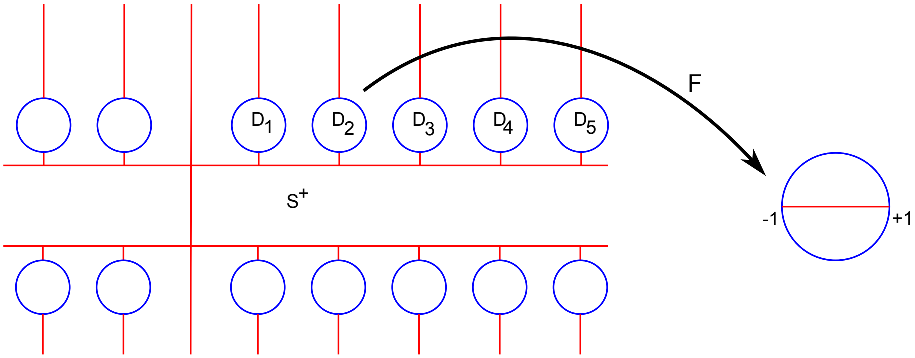


for some parameters $\boxed{d_n \xrightarrow[n \to +\infty]{} +\infty}$ and $\boxed{w_n \xrightarrow[n \to \infty]{} \frac{1}{2}}$.

Using Bishop's construction by quasiconformal foldings,
F may be extended to a quasiregular map $F : \mathbb{C} \to \mathbb{C}$ such that:

- $\forall z \in \mathbb{C},\ F(-z) = F(z)$ and $F(\overline{z}) = \overline{F(z)}$
- $\mathrm{Crit}(F) = \{-1, +1\} \cup \{w_n,\ n \geqslant 1\} \cup \left\{\frac{1}{2}\right\} \subset \overline{\mathbb{D}}$ with $w_n \xrightarrow[n \to +\infty]{} \frac{1}{2}$
- $\mathrm{Asym}(F) = \emptyset$



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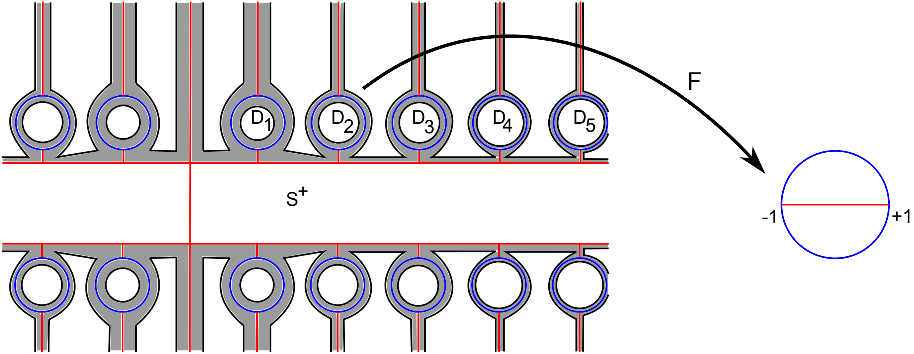
- $\mathrm{Asym}(F) = \emptyset$
- $\mathrm{supp}\,(F^*(\mu_0))$ is small enough



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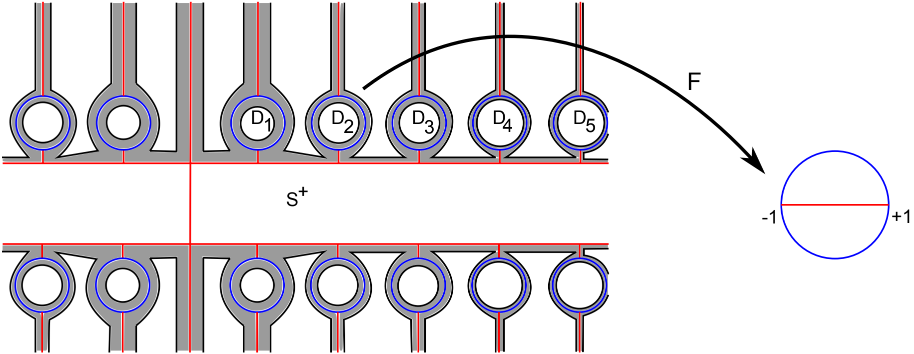
- $\mathrm{Asym}(F) = \emptyset$
- $\mathrm{supp}\,(F^*(\mu_0))$ is small enough in order that $\phi|_{S^+} \approx \mathrm{Id}_{S^+}$

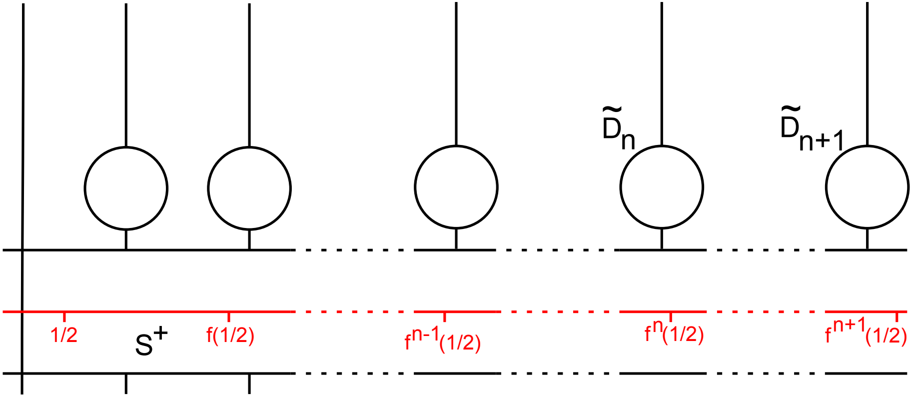


Let $f = F \circ \phi$ with $\phi : \mathbb{C} \to \mathbb{C}$ quasiconformal so that $\mu_{\phi^{-1}=F^*(\mu_0)}$.

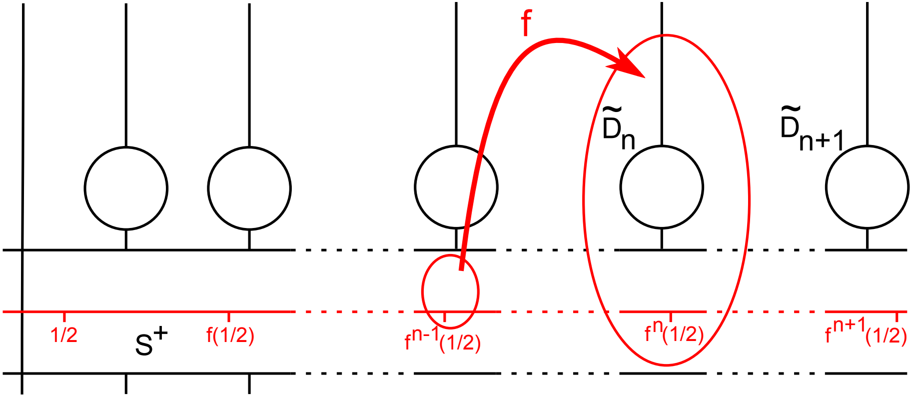
$\boxed{\text{Choice of the parameters } (\lambda, (d_n)_{n \geqslant 1}, (w_n)_{n \geqslant 1})}$

- $\boxed{\lambda > 0}$ is fixed so that $f^n\left(\frac{1}{2}\right) \xrightarrow[n \to +\infty]{} +\infty$ very fast.

$$\forall x \in \mathbb{R}, \quad f(x) = \cosh\left(\lambda \sinh\left(\phi|_{\mathbb{R}}(x)\right)\right) \approx \frac{1}{2} \exp\left(\frac{\lambda}{2} \exp(x)\right)$$



$\tilde{D}_n$
$\tilde{D}_{n+1}$
1/2
S^+
f(1/2)
$f^{n-1}(1/2)$
$f^n(1/2)$
$f^{n+1}(1/2)$



f
$\widetilde{D}_n$
$\widetilde{D}_{n+1}$
1/2
S^+
f(1/2)
$f^{n-1}(1/2)$
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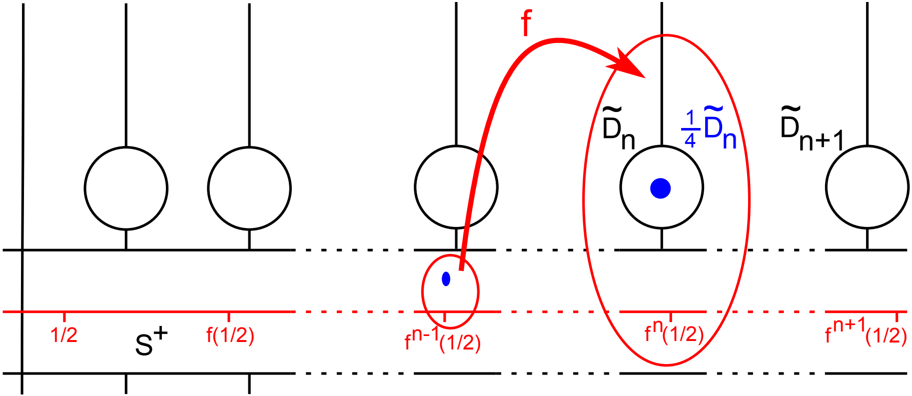
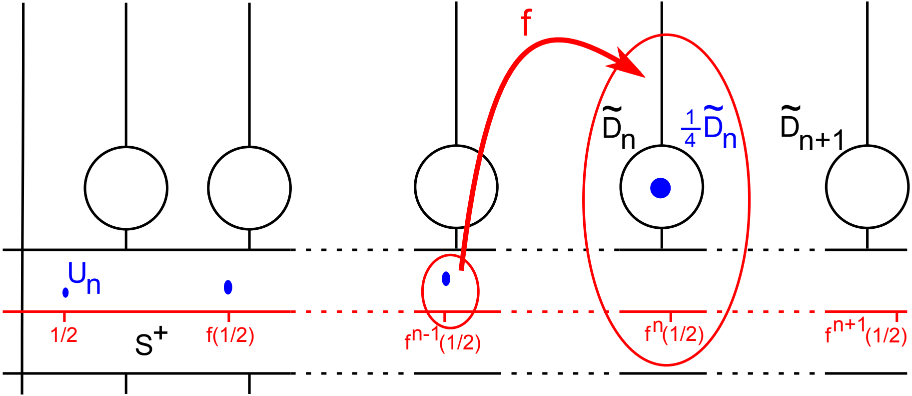
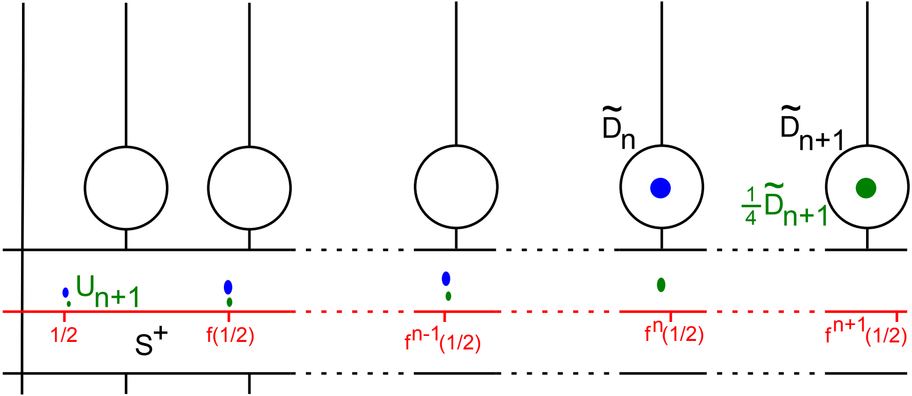


f
$\tilde{D}_n$
$\frac{1}{4}\tilde{D}_n$
$\tilde{D}_{n+1}$
1/2
S^+
f(1/2)
$f^{n-1}(1/2)$
$f^n(1/2)$
$f^{n+1}(1/2)$



$f^n(U_n) = \frac{1}{4}\widetilde{D}_n$ and $\mathrm{inradius}(U_n) \geqslant C.\left(\frac{df^n}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$



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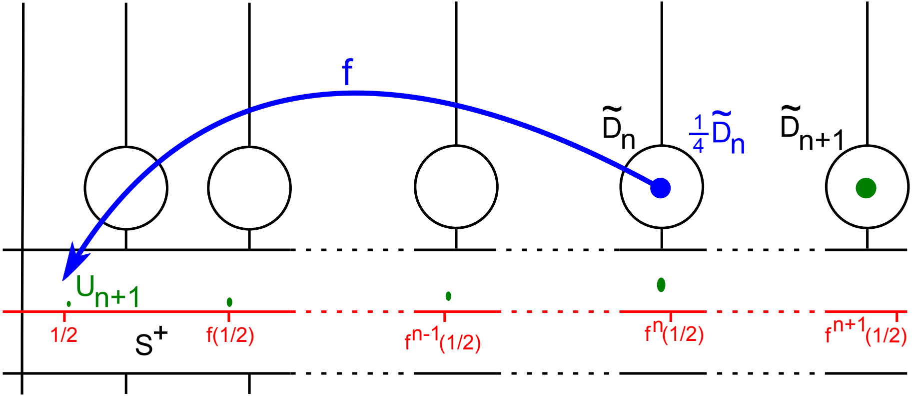
$$f^{n+1}(U_{n+1}) = \tfrac{1}{4}\widetilde{D}_{n+1} \quad \text{and} \quad \text{inradius}(U_{n+1}) \geqslant C.\left(\frac{df^{n+1}}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$$



$$f^{n+1}(U_{n+1}) = \tfrac{1}{4}\widetilde{D}_{n+1} \quad \text{and} \quad \text{inradius}(U_{n+1}) \geqslant C.\left(\frac{df^{n+1}}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$$

$$\widetilde{w}_n \in f\left(\tfrac{1}{4}\widetilde{D}_n\right) \quad \text{and} \quad \text{diam}\left(f\left(\tfrac{1}{4}\widetilde{D}_n\right)\right) \leqslant C'.\left(\tfrac{1}{4}\right)^{\widetilde{d}_n}$$

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- $\boxed{\lambda > 0}$ is fixed so that $f^n\left(\frac{1}{2}\right) \xrightarrow[n\to+\infty]{} +\infty$ very fast.
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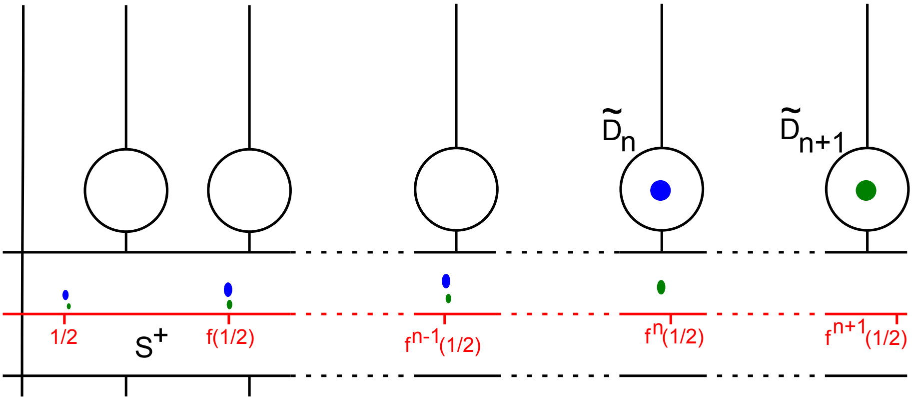
Therefore,

$$U_N \xrightarrow{f^{N+1}} U_{N+1} \xrightarrow{f^{N+2}} U_{N+2} \xrightarrow{f^{N+3}} U_{N+3} \xrightarrow{f^{N+4}} \dots$$

$\blacksquare$



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Fagella-G-Jarque, 2013

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Therefore W is eventually mapped into some U_n.

■

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Bishop's example has no unexpected wandering domains.

Corollary

$\exists$ two entire functions f, g (in $\mathcal{B}$) which both have no wandering domains but whose composition $f \circ g$ has some.

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Construct f, g like Bishop's example.

Chose the parameters $\boxed{\widetilde{d}_n \xrightarrow[n\to+\infty]{} +\infty}$ and $\boxed{\widetilde{w}_n \xrightarrow[n\to\infty]{} \frac{1}{2}}$ so that:

$$\forall k \geqslant N, \begin{cases} f^{4k+1}(U_{4k}) & \subset & U_{4k+1} \\ f^{4k+2}(U_{4k+1}) & \subset & U_{4k+2} \\ & & \\ & & \end{cases} \text{ and } \begin{cases} & & \\ & & \\ g^{4k+3}(U_{4k+2}) & \subset & U_{4k+3} \\ g^{4k+4}(U_{4k+3}) & \subset & U_{4k+4} \end{cases}$$

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$\blacksquare$