From a tree to a Persian carpet

Sébastien Godillon

Definition

$R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ rational map of degree $d \geqslant 2$

- The Julia set:

$$\mathcal{J}(R) = \overline{\{\text{repelling periodic points}\}}$$

- The Fatou set:

$$\mathcal{F}(R) = \widehat{\mathbb{C}} - \mathcal{J}(R) = \{\text{points with stable behavior}\}$$

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Proposition

$\mathcal{J}(R)$ is a fully invariant non-empty perfect compact set.
Furthermore

- either $\mathcal{J}(R)$ is connected,
- or else $\mathcal{J}(R)$ has uncountably many connected components.

Definition

A point $z \in \mathcal{J}(R)$ is said **buried** if

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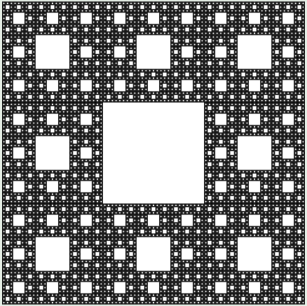
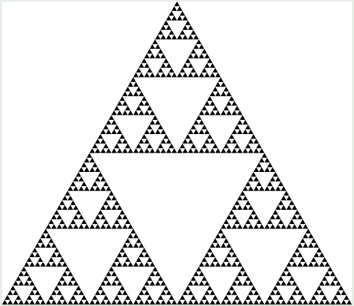
Examples (topological models)





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$$\forall \text{ Fatou component } F,\ z \notin \partial F$$

Question 0 Does there exist buried Julia components ?

Proposition

If P is a polynomial then

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Proposition

If the complement of every Julia component of $\mathcal{J}(R)$ is connected then there is no buried Julia component in $\mathcal{J}(R)$.

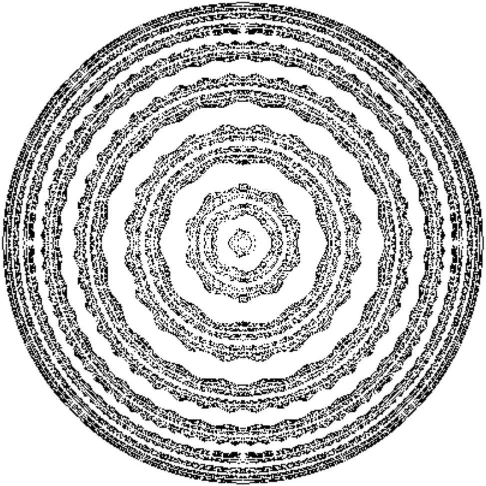
Theorem (McMullen 91)

$$g_{n,d,\varepsilon} : z \mapsto z^n + \frac{\varepsilon}{z^d}$$

If $|\varepsilon| > 0$ is small enough and if

$$\frac{1}{n} + \frac{1}{n} < 1 \qquad \text{(H0)}$$

then $\mathcal{J}(g_{n,d,\varepsilon})$ is a Cantor of Jordan curves.



A Cantor of Jordan curves

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Question 1 Does there exist buried Julia components for rational maps of degree $d < 5$?

Theorem (Pilgrim-Tan Lei 00)

If R is geometrically finite then
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Question 2 Does there exist buried Julia components
which are wandering points ?

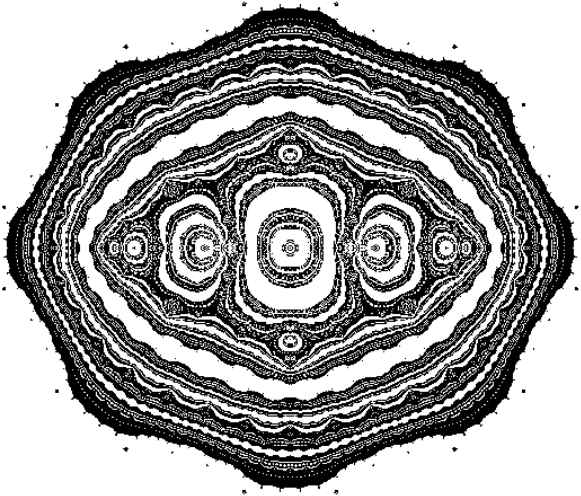
Question 3 Does there exist buried Julia components
which are neither a point nor a Jordan curve ?

Theorem

$$f_\varepsilon : z \mapsto \frac{(1-\varepsilon)\left[(1-4\varepsilon+6\varepsilon^2-\varepsilon^3)z-2\varepsilon^3\right]}{(z-1)^2\left[(1-\varepsilon-\varepsilon^2)z-2\varepsilon^2(1-\varepsilon)\right]}$$

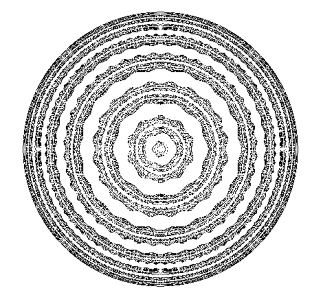
If $|\varepsilon| > 0$ is small enough then $\mathcal{J}(f_\varepsilon)$ contains buried Julia components of several types:

1. wandering Jordan curves
2. wandering (and preperiodic) points
3. preperiodic Julia components which are quasiconformally homeomorphic to a finite covering space of $\mathcal{J}\left(z \mapsto \frac{1}{(z-1)^2}\right)$



A Persian carpet

McMullen example: $g_{n,d,\varepsilon} : z \mapsto z^n + \frac{\varepsilon}{z^d}$



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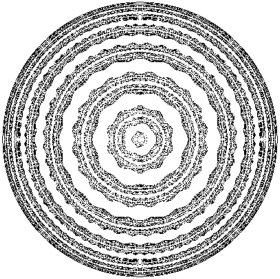

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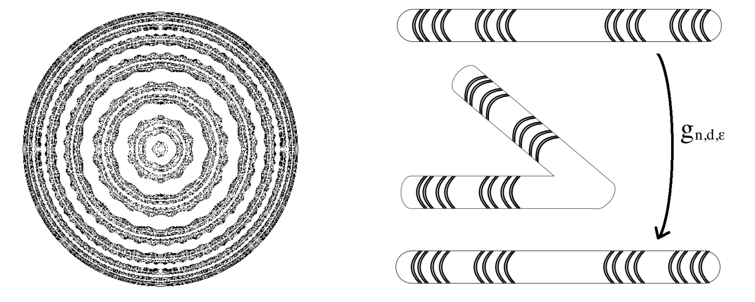

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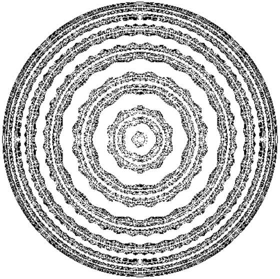
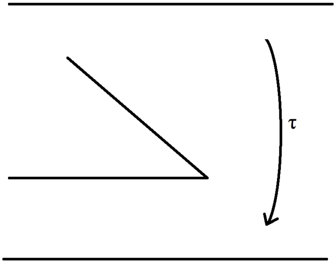




Theorem

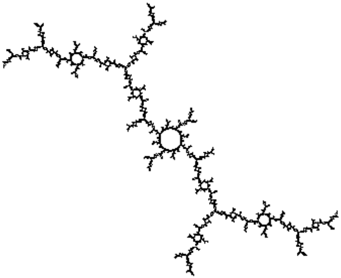
$\exists$ a continuous surjective map $\pi : \widehat{\mathbb{C}} \to T$ such that

$$\begin{tikzcd} \mathcal{J}(g_{n,d,\varepsilon}) \arrow[r, "g_{n,d,\varepsilon}"] \arrow[d, "\pi"'] & \mathcal{J}(g_{n,d,\varepsilon}) \arrow[d, "\pi"] \\ \mathcal{J}(T) \arrow[r, "\tau"'] & \mathcal{J}(T) \end{tikzcd}$$

where $\mathcal{J}(T) \subset T$ is a Cantor set and

$$\forall x \in \mathcal{J}(T),\ \pi^{-1}(x) \text{ is a Jordan curve Julia component}$$

$P : z \mapsto z^2 + c$ with $c \approx -0.157 + 1.032i$



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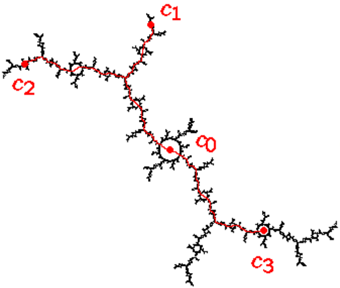
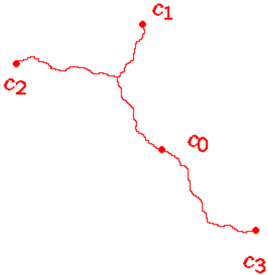
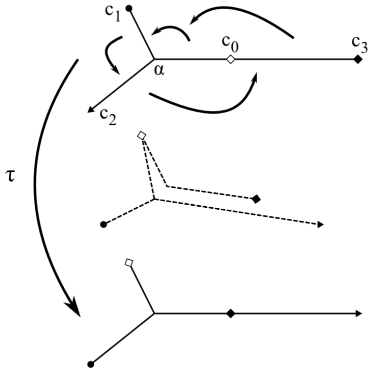
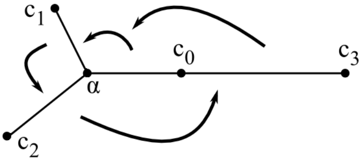


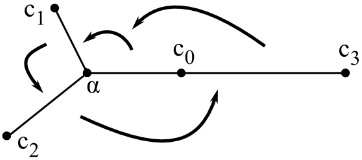


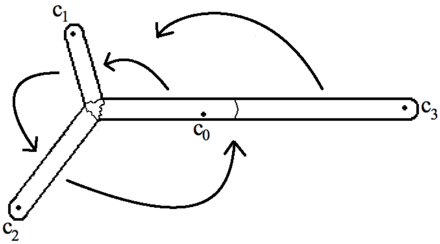
Let $\mathcal{H}$ be the associated Hubbard tree.

$(\mathcal{H}, P|_{\mathcal{H}})$ is conjugated to the dynamical tree (T, τ).

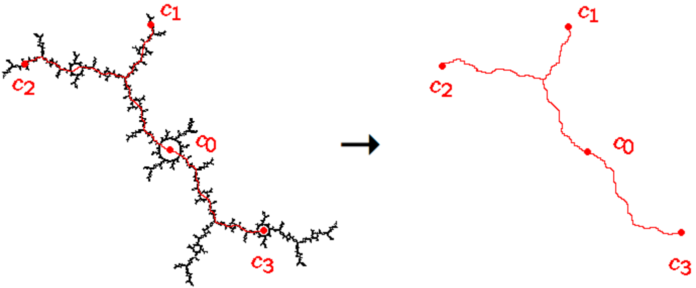



c_1
c_0
c_3
α
c_2


c_1
c_0
c_3
α
c_2


c_1
c_0
c_3
c_2

$P : z \mapsto z^2 + c$ with $c \approx -0.157 + 1.032i$



Let $\mathcal{J}(\mathcal{H})$ be the Cantor set $J(P) \bigcap \mathcal{H}$.

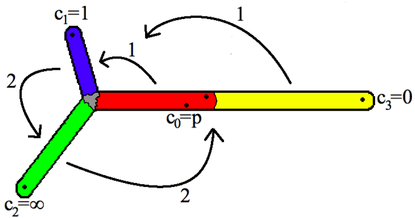
Theorem

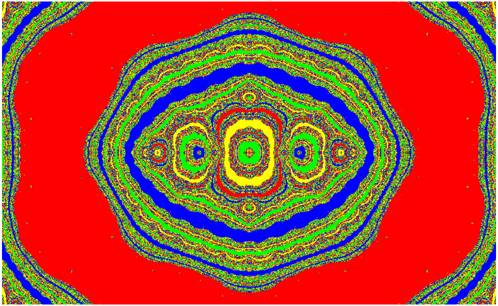
$\exists$ a continuous surjective map $\pi : \widehat{\mathbb{C}} \to \mathcal{H}$ and
$\exists$ an invariant subset $\mathcal{J}_{\mathcal{H}}(f_\varepsilon) \subset \mathcal{J}(f_\varepsilon)$ such that

$$\begin{array}{ccc} \mathcal{J}_{\mathcal{H}}(f_\varepsilon) & \xrightarrow{f_\varepsilon} & \mathcal{J}_{\mathcal{H}}(f_\varepsilon) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{J}(\mathcal{H}) & \xrightarrow[P|_{\mathcal{H}}]{} & \mathcal{J}(\mathcal{H}) \end{array}$$

where $\forall x \in \mathcal{J}(\mathcal{H})$, $\pi^{-1}(x)$ is a Julia component. Moreover

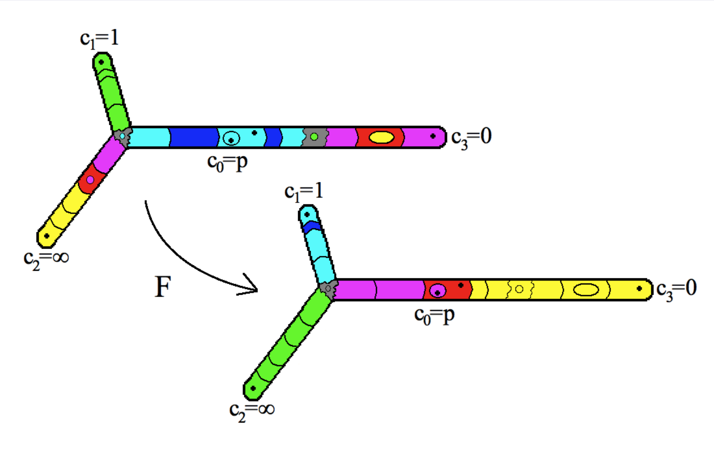
- $\pi^{-1}(\alpha)$ is fixed, buried and homeomorphic to $\mathcal{J}\left(z \mapsto \frac{1}{(z-1)^2}\right)$
- $\forall x \in \mathcal{J}(\mathcal{H}) - \bigcup_{n \geqslant 0} (P^{\circ n})^{-1}(\alpha)$, $\pi^{-1}(x)$ is a Jordan curve


$c_1=1$
1
1
2
$c_3=0$
$c_0=p$
2
$c_2=\infty$

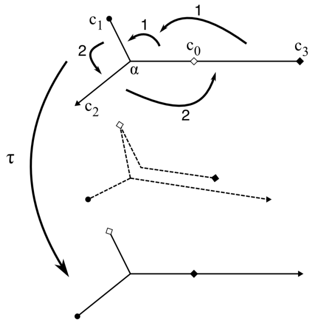


Proof

By quasiconformal surgery:



Proof: Branching point (Step 1)



Proof: Branching point (Step 1)

$$
\begin{tikzcd}
c_1 = 1 \arrow[r, "2"] & c_2 = \infty \arrow[r, "2"] & c_3 = 0 \arrow[ll, bend left, "1"]
\end{tikzcd}
$$

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$$\widehat{f} = \left(z \mapsto z^2\right) \circ \left(z \mapsto \frac{1}{1-z}\right) = \left(z \mapsto \frac{1}{(1-z)^2}\right)$$

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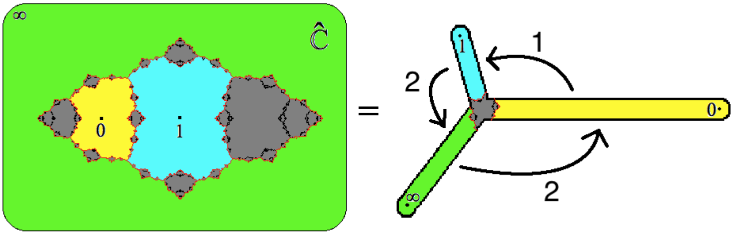
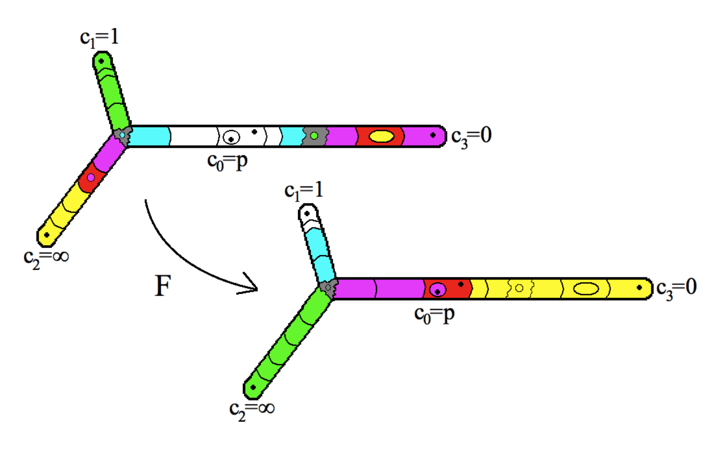
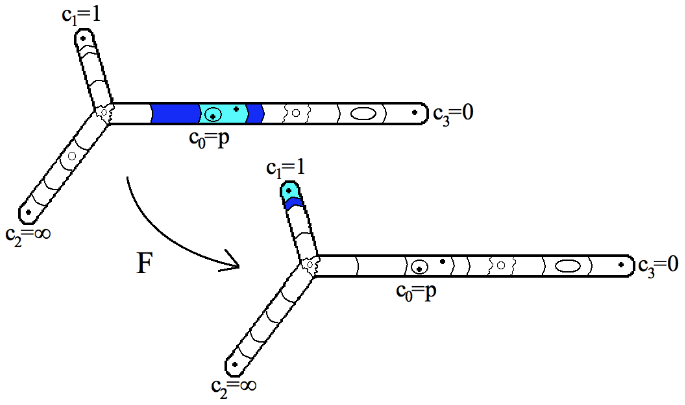


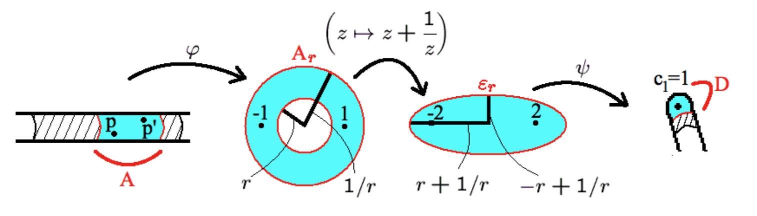
Proof: Cutting up (Step 2)



Proof: Folding point (Step 3)

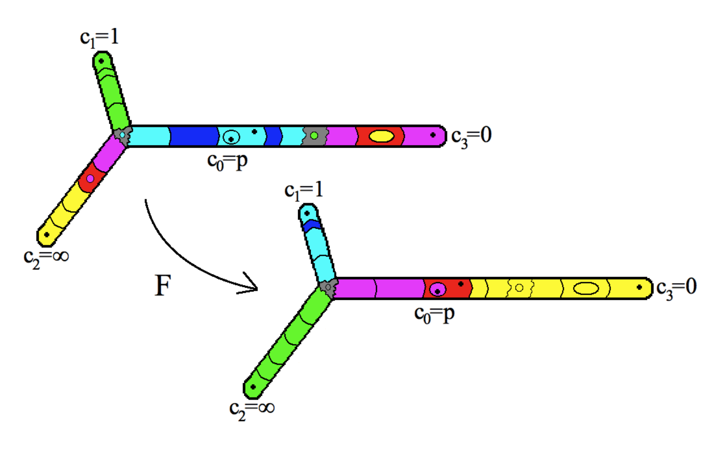


Proof: Folding point (Step 3)



$$F|_A = \psi \circ \left(z \mapsto z + \frac{1}{z}\right) \circ \varphi$$

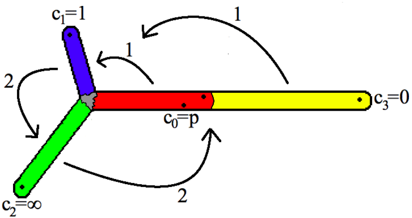
Proof: Uniformization (Step 4)



Proof: Uniformization (Step 4)

From Morrey-Ahlfors-Bers theorem,
$\exists$ a quasiconformal map ϕ such that

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[r, "F"] \arrow[d, "\phi"'] & \widehat{\mathbb{C}} \arrow[d, "\phi"] \\ \widehat{\mathbb{C}} \arrow[r, "f_\varepsilon"'] & \widehat{\mathbb{C}} \end{tikzcd}$$


c1=1
1
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2
c3=0
c0=p
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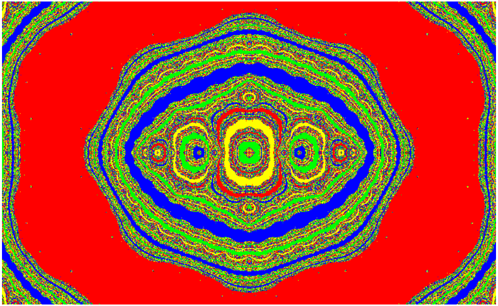
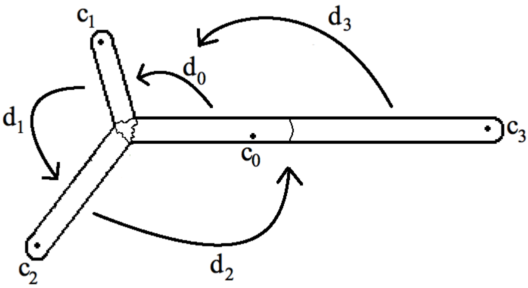
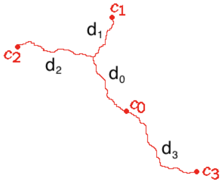
c2=∞

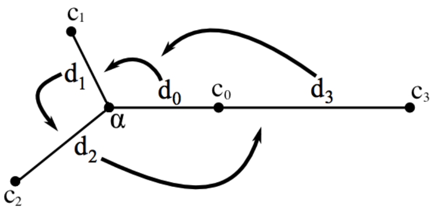


Generalized problem:



Provide the Hubbard tree $\mathcal{H}$ with a weight function w.





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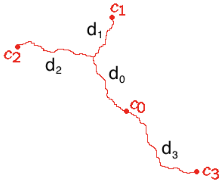
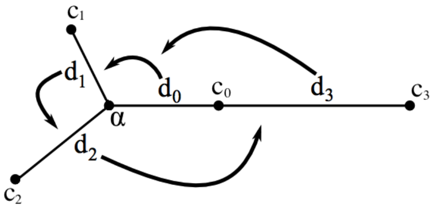
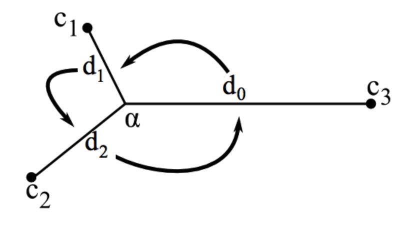




Question: Does there exist a rational map whose exchanging dynamics of Julia components is "encoded" by the weighted Hubbard tree $(\mathcal{H}, w)$?

Step 1: Branching point



Step 1: Branching point

$$\begin{tikzcd} c_1 \arrow[r, "d_1"] & c_2 \arrow[r, "d_2"] & c_3 \arrow[ll, bend left, "d_0"] \end{tikzcd}$$

Step 1: Branching point

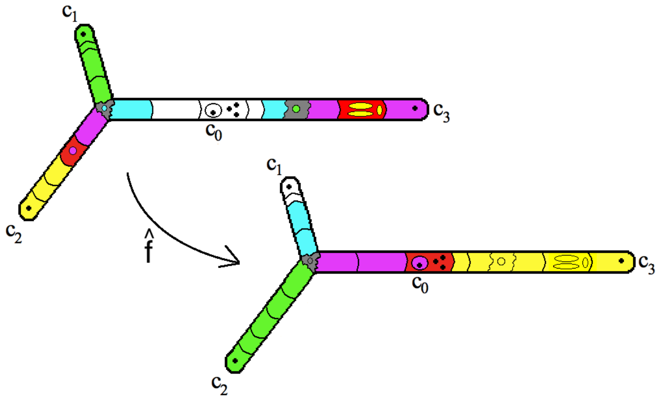
$$\begin{tikzcd} c_1 \arrow[r, "d_1"] & c_2 \arrow[r, "d_2"] & c_3 \arrow[ll, bend left, "d_0"] \end{tikzcd}$$

Lemma 1: Topological obstruction

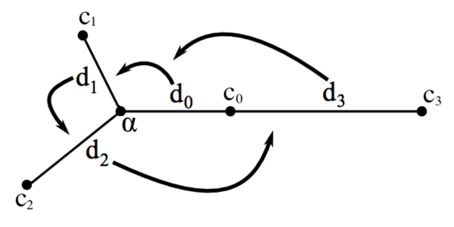
There exists a rational model $\widehat{f}$ if and only if

$$\begin{cases} \widehat{d} = \dfrac{1}{2}(d_0 + d_1 + d_2 - 1) \text{ is an integer } \geqslant 2 \\ \\ \max\{d_0, d_1, d_2\} \leqslant \widehat{d} \end{cases} \tag{H1}$$

Step 2: Cutting up



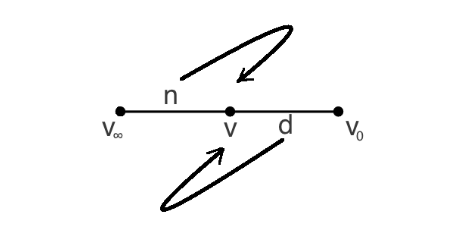
Definition (Thurston obstruction)



$$\left\{\begin{array}{lcl} \tau([\alpha, c_0]) & = & [\alpha, c_1] \\ \tau([\alpha, c_1]) & = & [\alpha, c_2] \\ \tau([\alpha, c_2]) & = & [\alpha, c_0] \cup [c_0, c_3] \\ \tau([c_0, c_3]) & = & [\alpha, c_1] \cup [\alpha, c_0] \end{array}\right. \rightarrow M_{\mathcal{H}} = \begin{pmatrix} 0 & \frac{1}{d_0} & 0 & 0 \\ 0 & 0 & \frac{1}{d_1} & 0 \\ \frac{1}{d_2} & 0 & 0 & \frac{1}{d_2} \\ \frac{1}{d_3} & \frac{1}{d_3} & 0 & 0 \end{pmatrix}$$

$(\mathcal{H}, w)$ is said not obstructed if $\lambda(M_{\mathcal{H}}) < 1$.

McMullen example

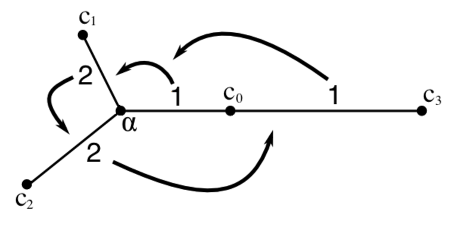


$$\left\{\begin{array}{rcl}\tau([v_\infty, v]) & = & [v_\infty, v] \cup [v, v_0] \\ \tau([v, v_0]) & = & [v_\infty, v] \cup [v, v_0]\end{array}\right. \to M_T = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} \\ \frac{1}{d} & \frac{1}{d} \end{pmatrix}$$

The McMullen example is not obstructed if and only if

$$\lambda(M_T) = \frac{1}{n} + \frac{1}{d} < 1 \tag{H0}$$

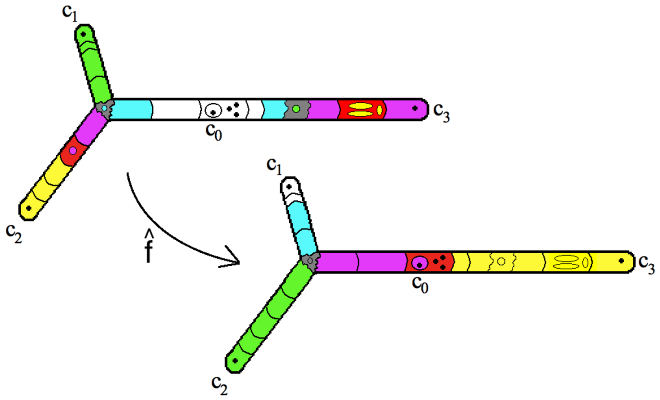
Persian carpet example



$$\left\{\begin{array}{rcl} \tau([\alpha, c_0]) & = & [\alpha, c_1] \\ \tau([\alpha, c_1]) & = & [\alpha, c_2] \\ \tau([\alpha, c_2]) & = & [\alpha, c_0] \cup [c_0, c_3] \\ \tau([c_0, c_3]) & = & [\alpha, c_1] \cup [\alpha, c_0] \end{array}\right. \rightarrow M_{\mathcal{H}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

In that case: $\lambda(M_{\mathcal{H}}) \approx 0.918 < 1$

Step 2: Cutting up



Step 2: Cutting up

Lemma 2: Analytical obstruction

There exists a system of equipotentials of $\widehat{f}$ if and only if

$$(\mathcal{H}, w) \text{ is not obstructed} \tag{H2}$$

Theorem

If the weighted Hubbard tree $(\mathcal{H}, w)$ satisfies

$$\begin{cases} \widehat{d} = \dfrac{1}{2}(d_0 + d_1 + d_2 - 1) \text{ is an integer } \geqslant 2 \\ \max\{d_0, d_1, d_2\} \leqslant \widehat{d} \end{cases} \tag{H1}$$

$$(\mathcal{H}, w) \text{ is not obstructed} \tag{H2}$$

then $\exists$ some rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d = \widehat{d} + d_3$ whose exchanging dynamics is "encoded" by $(\mathcal{H}, w)$.

Question: What happens for more sophisticated trees ?

Thank you for your attention!